CHAPTER

10

Consumer Surplus and Dead Weight Loss

This chapter brings together all the concepts from consumer theory — and in the process illustrates the difference between (uncompensated) demand and marginal willingness to pay (or compensated demand). We already developed (uncompensated) demand in Chapter 9 where we brought our A, B and C points from the consumer diagram into a new graph that had quantity on the horizontal and price on the vertical axis. But we only used the A and C points — with B playing no real role other than allowing us to see how big the substitution effect versus the income effect was. *Compensated* demand curves arise from *compensated* budgets, and thus we now turn our attention to point B. In particular, we see that the compensated demand curve connects A and B (rather than A and C) and that this curve can also be viewed as our *marginal willingness to pay* curve. (The same distinction between compensated and uncompensated curves can be made for the supply curves that emerge from the worker and saver diagrams — but we leave that until later (Chapter 19) in the text.) We then find that it is this marginal willingness to pay curve that we can use to measure consumer surplus and changes in consumer welfare, not the uncompensated demand curve from Chapter 9. The two curves are the same only in one very special case.

Chapter Highlights

The main points of the chapter are:

1. We can quantify in dollar terms the value people place on participating in a market — and we define this as the **consumer surplus**. We can similarly quantify the value people place on either getting a lower price or having to accept a higher price.

2. To do this, we need to know the **marginal willingness to pay** for each of the goods a consumer consumes — and this is closely related to the changing MRS along the indifference curve on which the consumer operates.

3. The **marginal willingness to pay** is derived from a single indifference curve — and thus **incorporates only substitution effects**. It is the **same as the uncompensated demand curve only if there are no income effects** — only if tastes are quasilinear in the good we are modeling.

4. **Price-distorting taxes (and subsidies) are inefficient** in the consumer model **to the extent to which they give rise to substitution effects**. Therefore, the inefficiency goes away if the degree of substitutability between goods is zero.

5. The **deadweight loss** from taxes (or subsidies) can be measured as a **distance in the consumer diagram** or as **an area along the marginal willingness to pay curve**.

6. To say that a policy is **inefficient** is the same as to say that there exists in principle a way to compensate those who lose from the policy with the winnings from those who gain. That is not the same as saying that such a policy exists in practice.

7. If you are reading the B-part of the chapter: the **compensated (Hicksian) demand function** is derived from the expenditure minimization problem while the uncompensated budget is derived from the utility maximization problem. The **Slutzky equation** then illustrates the relationship of the slope of the uncompensated demand curve to the slope of the compensated demand (or marginal willingness to pay) curve. This is one of several links — summarized in our **duality** picture at the end of the chapter — between concepts emerging from the utility maximization and the expenditure minimization problems.

Using the LiveGraphs

For an overview of what is contained on the LiveGraphs site for each of the chapters (from Chapter 2 through 29) and how you might utilize this resource, see pages 2-3 of Chapter 1 of this *Study Guide*. To access the LiveGraphs for Chapter 10, click the *Chapter 10* tab on the left side of the LiveGraphs web site.

We have not at this point developed *Exploring Relationships* modules for this chapter, and thus the LiveGraphs for Chapter 10 are limited to the *Animated Graphics*, *Static Graphics* and *Downloads*. However, if you have not reviewed the *Exploring Relationships* module from Chapter 9, you might want to do so now. The module shows clearly how point B emerges from the substitution effect and how the uncompensated demand curve has nothing to do with B unless the good happens to be borderline between normal and inferior — i.e. unless the good happens to be quasilinear.

10A Solutions to Within-Chapter Exercises for Part A

Exercise 10A.1 *As a way to review material from previous chapters, can you identify assumptions on tastes that are* sufficient *for me to know for sure that my indifference curve will be tangent to the budget line at the optimum?*

Answer: This will be true so long as, in addition to the usual assumptions about tastes, we assume that both goods are "essential" as defined in Chapter 5. This implies that indifference curves never cross the axes — and thus corner solutions are not possible.

Exercise 10A.2 *Demonstrate that own-price demand curves are the same as marginal willingness to pay curves for goods that can be represented by quasilinear tastes.*

Answer: This is easy to see once you realize that points B and C in the lower panel of Graph 10.2 in the text will lie on top of one another when there are no income effects — i.e. when tastes are quasilinear in gasoline. C lies to the left of B when gasoline is normal, to the right when gasoline is inferior — so the only time they lie exactly on top of one another is if gasoline is borderline normal/inferior.

Exercise 10A.3 *Using the graphs in Graph 9.2 of the previous chapter, determine under what condition own price demand curves are steeper and under what conditions they are shallower than marginal willingness to pay curves.*

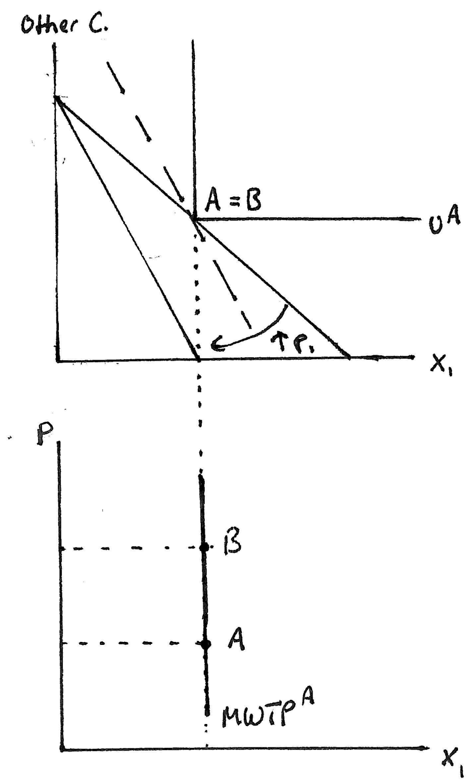
Answer: Once we realize that $MWTP$ curves connect A and B in the lower panels of the Graph 9.2 in the chapter, it is easy to see the relationship by simply connecting these and forming the $MWTP$ curves. In panel (a) where x is a normal good, $MWTP$ is steeper than own-price demand; in panel (b) where x is (regular) inferior, $MWTP$ is shallower than own-price demand.

Exercise 10A.4 *What does the $MWTP$ or compensated demand curve look like if the two goods are perfect complements?*

Answer: When the two goods are perfect complements, the entire change in behavior from a price change is an income effect — with no substitution effect. Since $MWTP$ curves only incorporate substitution effects, this implies that the $MWPT$ curve has to be perfectly vertical. This is illustrated in Graph 10.1 (next page).

Exercise 10A.5 *How would Graph 10.4a change if x_1 were an inferior rather than a normal good?*

Answer: Bundle B would then lie to the right rather than the left of A — causing the own-price demand curve corresponding to the lower income (I^B) to lie to the



Graph 10.1: $MWTP$ for Perfect Complements

right of the demand curve corresponding to the higher income (I^A). When a good is inferior, the demand curve therefore shifts out when income falls and in when income increases, the reverse of what is true when the good is normal.

Exercise 10A.6 *How would Graph 10.4b change if x_1 were inferior rather than normal?*

Answer: If x_1 is inferior, then B must again lie to the *right* of A (just as in the previous exercise). Thus, the marginal willingness to pay curve corresponding to higher utility (i.e. u^A) must lie to the left of the $MWTP$ curve corresponding to lower utility (i.e. u^B).

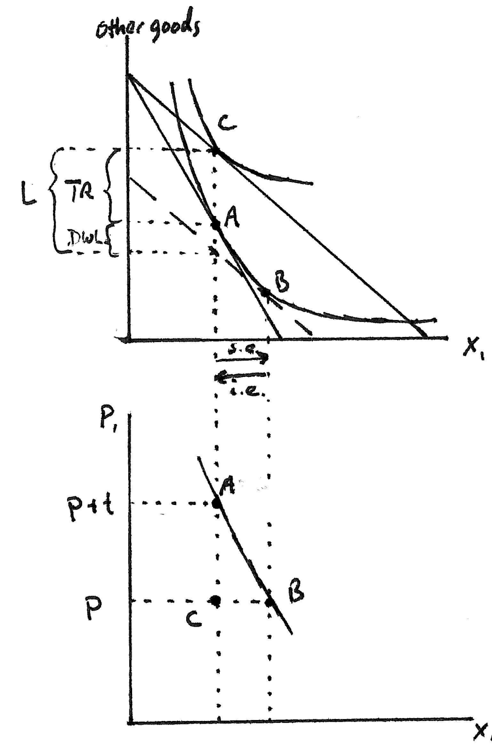
Exercise 10A.7 *On the lower panel of Graph 10.5b, where does the $MWTP$ curve corresponding to the indifference curve that contains bundle C lie?*

Answer: It lies exactly on top of the $MWTP(u^A)$ curve that is depicted in the graph. Since there are no income effects, every $MWTP$ curve must lie on the uncompensated demand curve that now incorporates only substitution effects (because x_1 is quasilinear).

Exercise 10A.8 *How do the upper and lower panels of Graph 10.5a change when gasoline is an inferior good?*

Answer: If gasoline is an inferior good, then C lies to the right (instead of the left) of B in the top graph — which causes it to lie to the right (instead of left) of B in the lower panel. This implies that the uncompensated demand curve that connects A and C is now steeper (rather than shallower) than the compensated demand (or $MWTP$) curve that connects A and B.

Exercise 10A.9 *Can you think of a scenario under which a consumer does not change her consumption of a good when it is taxed but there still exists an inefficiency from taxation?*



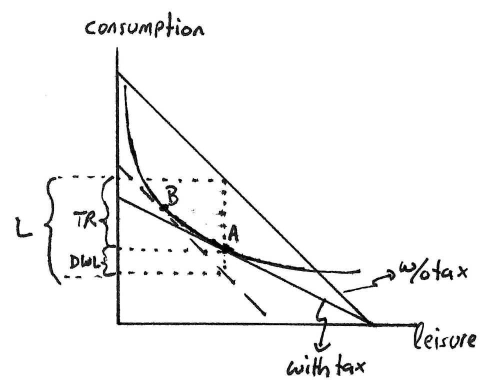
Graph 10.2: DWL without a Change in Behavior

Answer: This would require the existence of a substitution effect (which causes the inefficiency) that is exactly offset by an income effect. This is illustrated in Graph 10.2 (on the previous page) and occurs if a good was borderline regular inferior/Giffen. In the top panel, A represents the bundle consumed under the higher (tax-inclusive) price while bundle C represents the bundle consumed at the lower (before-tax) price. The two bundles contain the same quantity of x_1 — so behavior relative to consumption of the taxed good does not change. However, there is clearly an underlying substitution effect that is masked by an offsetting income effect. This substitution effect causes the tax revenue (labeled TR) to be less than what could have been raised by a lump sum tax that makes the consumer no worse off (labeled L). The difference between the two is the deadweight loss.

The lower panel of the graph illustrates the relevant compensated demand curve along which this deadweight loss can be measured (as described in the next section.) Note that the uncompensated demand curve would connect A and C — and would be perfectly vertical with no change in behavior.

Exercise 10A.10 *On a graph with consumption on the vertical axis and leisure on the horizontal, illustrate the deadweight loss of a tax on all consumption (other than the consumption of leisure).*

Answer: This is illustrated in Graph 10.3 where A is the consumption bundle after the tax is imposed and TR is the tax revenue collected from the tax. Were we to employ a non-distortionary tax instead, we could shift the before-tax budget in parallel all the way to the tangency at B and make the consumer worse off. Through such a lump-sum tax, we could raise L in revenue — more than we raise under the distortionary tax. The difference between L and TR is the deadweight loss DWL.



Graph 10.3: DWL from a tax on all Consumption

Exercise 10A.11 *Using Graph 10.8a, verify that the relationship between own price demand and marginal willingness to pay is as depicted in panels (a) through (c) of Graph 10.9.*

Answer: In order to plot the uncompensated demand curve into the graph that has the compensated demand (or $MWTP$) curve, all we have to do is determine where C lies — i.e. where would the consumer consume in the absence of the distortionary and the lump sum tax. Starting at B, this simply means asking where the consumer would consume if she had more income (because the budget that is tangent at B is parallel to the no-tax budget). If h is a normal good, then such an increase in income implies more consumption of h — which would put C to the right of B. If h is quasilinear, then additional income does not affect consumption of h and would thus put C at the same level of h as B. Finally, if housing is inferior, an increase in income causes less consumption of h — thus causing C to lie to the left of B. This verifies the placement of C relative to B in the three panels of the graph in the text.

Exercise 10A.12 *The two proposals also result in different levels of tax revenue. Which proposal actually results in higher revenue for the government? Does this strengthen or weaken the policy proposal "to broaden the base and lower the rates"?*

Answer: The tax proposal that imposes $2t$ on the single family housing market results in tax revenues equal to $2t$ times q_{p+2t}. The alternative proposal raises t times q_{p+t} in each market, or $2t$ times q_{p+t}. Thus, tax revenue under the latter proposal is a bit larger, t times $(q_{p+t} - q_{p+2t})$ to be exact. The proposal with less deadweight loss therefore also produces more revenue — which strengthens the advice to broaden the base and lower the rates.

10B Solutions to Within-Chapter Exercises for Part B

Exercise 10B.1 *In Graph 10.5b we illustrated that $MWTP$ curves and own price demand curves are the same when tastes are quasilinear. Suppose tastes can be modeled with the quasilinear utility function $u(x_1,x_2) = \alpha \ln x_1 + x_2$. Verify a generalization of the intuition from Graph 10.5b — that demand functions and compensated demand functions are identical for x_1 in this case.*

Answer: Solving the utility maximization problem

$$\max_{x_1,x_2} \alpha \ln x_1 + x_2 \text{ subject to } p_1x_1 + p_2x_2 = I, \tag{10.1}$$

we get the (uncompensated) demand function $x_1 = \alpha p_2/p_1$. We get exactly the same function when we solve the expenditure minimization problem

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } \alpha \ln x_1 + x_2 = u. \tag{10.2}$$

Exercise 10B.2 *Verify the solutions given in equations (10.8).*

Answer: Solving the maximization problem

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } x_1^{\alpha}x_2^{(1-\alpha)} = u, \tag{10.3}$$

we get first order conditions

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \alpha x_1^{\alpha-1}x_2^{(1-\alpha)} = 0,\\ \frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - (1-\alpha)x_1^{\alpha}x_2^{-\alpha} = 0.\end{aligned} \tag{10.4}$$

Solving these for x_1, we get

$$x_1 = \frac{\alpha p_2 x_2}{(1-\alpha)p_1}. \tag{10.5}$$

Plugging this into the constraint $u = x_1^{\alpha}x_2^{(1-\alpha)}$, we get

$$u = \left(\frac{\alpha p_2 x_2}{(1-\alpha)p_1}\right)^{\alpha} x_2^{(1-\alpha)} = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{\alpha} x_2. \tag{10.6}$$

Solving this equation for x_2, we then get

$$x_2 = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u \tag{10.7}$$

which is then equal to $h_2(p_1, p_2, u)$ as derived in the text. Substituting this back into (10.5), we also get

$$x_1 = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u = h_1(p_1, p_2, u). \tag{10.8}$$

Exercise 10B.3 *Verify the solutions given in equations (10.8) and (10.9).*

Answer: Plugging the demand functions for x_1 and x_2 into the Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha}x_2^{(1-\alpha)}$, we get

$$\left(\frac{\alpha I}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)I}{p_2}\right)^{(1-\alpha)} = \frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha}p_2^{(1-\alpha)}} \tag{10.9}$$

which is equal to the indirect utility function $V(p_1, p_2, I)$.

Similarly, multiplying the compensated demand functions by prices and adding them, we get

$$p_1\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)} u + p_2\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u = \frac{u p_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}} \tag{10.10}$$

which is equal to the expenditure function $E(p_1, p_2, u)$.

Exercise 10B.4 *Verify that (10.11) and (10.12) are true for the functions that emerge from utility maximization and expenditure minimization when tastes can be modeled by the Cobb-Douglas function* $u(x_1,x_2)=x_1^{\alpha}x_2^{(1-\alpha)}$.

Answer: Plugging the expenditure function into the (uncompensated) demand function for x_1, we get

$$x_1(p_1,p_2,E(p_1,p_2,u))=\frac{\alpha}{p_1}\left(\frac{up_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)$$
$$=\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)}u=h_1(p_1,p_2,u). \tag{10.11}$$

The same holds if we substitute the expenditure function into the (uncompensated) demand function for x_2.

Also, if we substitute the indirect utility function into the compensated demand function for x_1, we get

$$h_1(p_1,p_2,V(p_1,p_2,I))=\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{(1-\alpha)}\left(\frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha}p_2^{(1-\alpha)}}\right)$$
$$=\frac{\alpha I}{p_1}=x_1(p_1,p_2,I) \tag{10.12}$$

and again the same holds if we substitute the indirect utility function into the compensated demand function for x_2.

Exercise 10B.5 *Verify that the equations in (10.19) are correct for the Cobb-Douglas utility function* $u(x_1,x_2)=x_1^{\alpha}x_2^{(1-\alpha)}$.

Answer: We previously derived the demand function for x_1 under these tastes as $x_1(p_1,p_2,I)=\alpha I/p_1$. From this, we can derive

$$T=tx_1(p_1+t,p_2,I)=\frac{t\alpha I}{p_1+t}. \tag{10.13}$$

We also previously derived the indirect utility and expenditure functions for these tastes as

$$V(p_1,p_2,I)=\frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{p_1^{\alpha}p_2^{(1-\alpha)}} \text{ and } E(p_1,p_2,u)=\frac{up_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \tag{10.14}$$

The lump sum tax L is, as just derived in the text, $L=I-E(p_1,p_2,V(p_1+t,p_2,I))$. Put in terms of the expressions derived previously, this implies

$$L=I-\left(\frac{p_1^{\alpha}p_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)\left(\frac{I\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p_1+t)^{\alpha}p_2^{(1-\alpha)}}\right)=I-I\left(\frac{p_1}{p_1+t}\right)^{\alpha}. \tag{10.15}$$

Exercise 10B.6 *Verify that the numbers calculated in the previous paragraph are correct.*

Answer: Plugging in the appropriate values, we get

$$T = \frac{t\alpha I}{p_1 + t} = \frac{(2.5)(0.25)(100,000)}{10+2.5} = 5,000 \quad (10.16)$$

and

$$L = I - I\left(\frac{p_1}{p_1+t}\right)^{\alpha} = 100,000 - 100,000\left(\frac{10}{10+2.5}\right)^{0.25} = 5,425.84. \quad (10.17)$$

Subtracting T from L gives us a dead weight loss of approximately \$426.

Exercise 10B.7 *What shape must the indifference curves have in order for the second derivative of the expenditure function with respect to price to be equal to zero (and for the "slice" of the expenditure function in panel (b) of Graph 10.14 to be equal to the blue line)?*

Answer: The indifference curves would have a sharp kink — as those for perfect complements. This is because it is the substitution effect that is creating the concavity of the expenditure function slice in the graph — and that strict concavity goes away only when the substitution effect goes away. And the substitution effect only goes away if the curvature of the indifference curve goes away. Put differently, the "naive" expenditure function that says that consumer will choose the same bundle to reach the same indifference curve when prices change is literally correct if the consumer does not substitute — i.e. if the goods are perfect complements.

Exercise 10B.8 *In a 2-panel graph with the top panel containing an indifference curve and the lower panel containing a compensated demand curve for x_1 derived from that indifference curve, illustrate the case when the inequality in equation (10.39) becomes an equality? (*Hint*: Remember that our graphs of compensated demand curves are graphs of the* inverse *of a slice of the compensated demand functions, with a slope of 0 turning into a slope of infinity.)*

Answer: The compensated demand curve will be perfectly vertical (which is equivalent to saying $\partial h_i(p_1, p_2, u)/\partial p_i = 0$) if and only if there are no substitution effects — i.e. if the indifference curve has a sharp kink as in the case of perfect complements.

End of Chapter Exercises

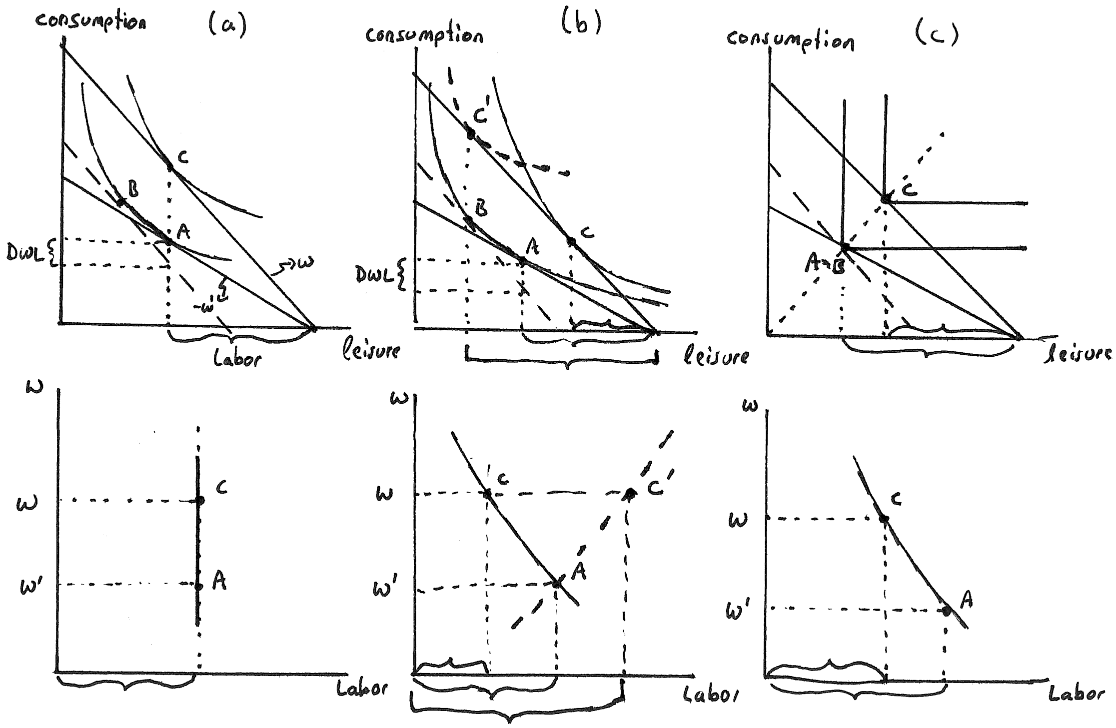
Exercise 10.2

Suppose that both consumption and leisure are always normal goods. Keep in mind the underlying cause for deadweight losses from wage-distoring taxation as you answer the questions below.

A: *Explain why the following either cannot happen or, if you think it can happen, how:*

(a) *Labor supply is perfectly vertical but there is a significant dead weight loss from taxing wages.*

Answer: In panel (a) of Graph 10.4, we illustrate a high wage w and a low (after-tax) wage w'. Tastes are drawn such that the optimal amount of leisure is the same under both wages — which translates in the lower graph to a vertical labor supply curve. However, there is a substitution effect that gives rise to a dead weight loss that is labeled DWL on the vertical axis in the upper graph. (In Chapter 19, we will show how to illustrate this in terms of the lower graph.)



Graph 10.4: Labor Supply and Deadweight Loss from Taxation

(b) *Labor supply is perfectly vertical and there is no dead weight loss from taxing wages.*

Answer: The only way this could happen is if the indifference curves in panel (a) are "tangent" at the same points A and C but the indifference curve that is tangent at A has a sufficient kink to cause $B = A$ — i.e. to eliminate a substitution effect. In principle, this is possible if the kink is not at a right angle (as in the case of perfect complements) — it has to be less sharp than that, but sufficiently sharp to eliminate the substitution effect.

(c) *Labor supply is downward sloping and there is a deadweight loss from taxation of wages.*

Answer: This is illustrated in panel (b) of Graph 10.4. Here, the consumer demands more leisure at the higher wage at bundle C than at the lower wage at bundle A — which results in a downward slope of the labor supply curve. Since there is a substitution effect similar to the one in panel (a), we still have the deadweight loss from the wage tax.

(d) *Labor supply is upward sloping and there is a deadweight loss from taxing wages.*

Answer: This can also happen — all we would have to change in panel (b) of the graph is to change the optimal indifference curve at the high wage to the dashed curve that is tangent at C'. This would result in less leisure demand at the higher wage than at the lower wage

— which results in an upward sloping labor supply curve. Since we have not changed the optimal indifference curve at the lower wage, the substitution effect remains, as does the deadweight loss.

(e) *Labor supply is downward sloping and there is no deadweight loss from taxing wages.*

Answer: This is illustrated in panel (c) of Graph 10.4. The substitution effect that causes the deadweight loss is eliminated by the assumption that tastes over consumption and leisure are now perfect complements. At the higher wage, the worker takes more leisure and thus works less — giving rise to a downward sloping labor supply curve.

(f) *Labor supply is upward sloping and there is no deadweight loss from taxing wages.*

Answer: In principle, this is still possible. We would still need a kink at the optimal after-tax point A — a kink that is sufficiently large to eliminate the substitution effect. Making this kink less sharp than it is in panel (c) would give us some room on the pre-tax budget to locate C to the left of $A = B$. That would give us an upward slope of the labor supply curve while preserving the absence of a substitution effect that causes the deadweight loss.

B: *Now suppose that tastes can be summarized by the CES utility* $u(c,\ell) = (0.5c^{-\rho} + 0.5\ell^{-\rho})^{-1/\rho}$, *where c is consumption and* ℓ *is leisure.*

(a) *Are there values for* ρ *that would result in the scenario in A(a)*

Answer: For the labor supply to be perfectly vertical, it would have to be the case that the optimal amount of leisure does not depend on the wage rate. To solve for the optimal amount of leisure, we solve the problem

$$\max_{c,\ell} = (0.5c^{-\rho} + 0.5\ell^{-\rho})^{-1/\rho} \text{ subject to } w(L-\ell) = c \tag{10.18}$$

where L is the leisure endowment. This gives us

$$\ell = \frac{L}{1 + w^{-\rho/(\rho+1)}}. \tag{10.19}$$

When ρ is set to zero, this reduces to $\ell = L/2$ — which implies leisure demand becomes independent of the wage rate and tastes are Cobb-Douglas. This further implies that labor supply becomes independent of the wage rate — i.e. the labor supply curve is vertical. The reason for this is that the substitution effect is exactly offset by the wealth effect, but the substitution effect still gives rise to a deadweight loss from wage taxation. Thus, the scenario in A(a) arises when $\rho = 0$.

(b) *Are there values for* ρ *that would result in the scenario in A(b)?*

The scenario in A(b), on the other hand, cannot arise — because the only way we get a perfectly vertical labor supply curve under these CES tastes is if $\rho = 0$ — but this implies the elasticity of substitution is 1 which introduces the substitution effect that causes the deadweight loss.

(c) *Are there values for* ρ *that would result in the scenario in A(c)?*

Answer: In order for labor supply to slope down, it has to be the case that leisure increases as the wage increases. We calculated the optimal leisure amount in equation (10.19). To see how this responds to an increase in the wage, we can take the derivative with respect to wage to get

$$\frac{\partial \ell}{\partial w} = \frac{\rho}{\rho+1}\left(\frac{L}{\left(1 + w^{-\rho/(\rho+1)}\right)^2 w^{(2\rho+1)/(\rho+1)}}\right). \tag{10.20}$$

In order for leisure to increase as the wage increases, this derivative has to be positive. The term in parentheses is positive (since $L > 0$ and $w > 0$) — which implies that the whole derivative is positive if and only if $\rho > 0$. Thus, as long as $\rho > 0$, the labor supply curve is downward sloping. And, as long as $\rho \neq \infty$ (which would put the elasticity of substitution at zero and eliminate substitution effects), there will be a deadweight loss. The scenario in A(c) therefore arises for $0 < \rho < \infty$.

(d) *Are there values for ρ that would result in the scenario in A(d)?*

Answer: The only way that labor supply is upward sloping is if leisure falls with an increase in the wage. We can therefore again look at the derivative of optimal leisure demand, which we calculated in equation (10.20). In order for leisure demand to fall as wage increases, this derivative has to be negative. And, since the term in parenthesis is positive, this happens if and only if $-1 < \rho < 0$. We furthermore know that the elasticity of substitution is positive so long as $\rho \neq \infty$ — which means it is positive when ρ falls between 0 and -1. Thus, the scenario in A(d) arises for $-1 < \rho < 0$.

(e) *Are there values for ρ that would result in the scenario in A(e) and A(f)?*

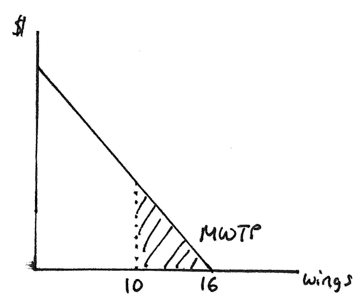
Answer: In order for there to be no deadweight loss, it has to be that there are no substitution effects. The only way to eliminate the substitution effect under the CES utility in this problem is for $\rho = \infty$. And, as ρ approaches ∞, the leisure demand from equation (10.19) converges to $\ell = L/(1+w^{-1})$ which has a positive derivative with respect to w. Thus, when $\rho = \infty$, leisure demand increases with increases in the wage — which implies labor supply decreases with an increase in the wage; i.e. labor supply slopes down. Thus the scenario in A(e) arises when $\rho = \infty$ but the scenario in A(f) cannot arise.

Exercise 10.6: *Ordering Appetizers*

Everyday Application: *Ordering Appetizers: I recently went out to dinner with my brother and my family. We decided we wanted wings for an appetizer and had a choice of getting 10 wings for $4.95 or 20 wings for $7.95. I thought we should get 10; my brother thought we should get 20 and prevailed.*

A: *At the end of the meal, we noticed that there were 4 wings left. My brother then commented: "I guess I am vindicated — it really was the right decision to order 20 rather than 10 wings."*

(a) *Is this a correct assessment; i.e. is the evidence of 4 wings at the end of the meal sufficient to conclude that my brother was right?*



Graph 10.5: Wings Left Over

Answer: It is not necessarily a correct assessment. What we know from the evidence that 4 wings were left at the end is that the $MWTP$ curve must cross the horizontal intercept at approximately 16 wings — i.e. the marginal value of additional wings is zero since we could have eaten them without paying anything more but chose not to. (Whether we left the wings or not, our total bill was the same since restaurants don't refund us for food we leave behind.) The second 10 wings cost us $3 — so in order for my brother to be correct, it must be that the total value we received from the 6 additional wings we ate was at least $3. That total value is represented as the shaded triangle in Graph 10.5 — and without knowing the height of that triangle, we can't be sure what the area of the triangle is. The height of the triangle is the marginal willingness to pay for the 10th wing. (Even if we did know that, we still would not know the shape of the $MWTP$ curve in between 10 and 16 — so even then we could not really be sure unless we assumed the linear shape that is depicted in the graph).

(b) *What if no wings were left at the end of the meal?*

Answer: Even if no wings had been left, I cannot interpret this as evidence that we in fact got more than $3 of value from the additional 10 wings. Before, we at least knew that the marginal value of the 17th wing was zero — now we do not know the marginal value of any of the wings. We might, for instance, have eaten them because we valued them each at one cent — which would mean we only got 10 cents of enjoyment from them. Or we might have valued them each at one dollar, in which case we would have gotten $10 worth of enjoyment.

(c) *What if 10 wings were left?*

Answer: Now we could be sure that my brother was wrong — because we would know that the marginal value of the 11th wing (and each one after that) was zero.

(d) *In order for us to leave wings on the table, which of our usual assumptions about tastes must be violated?*

Answer: Monotonicity — or at least the strict version of monotonicity. We could have eaten additional wings for free but chose not to. More is not better — just at least as good.

B: *Suppose that our $MWTP$ for wings (x) can be approximated by the function $MWTP = A - \alpha x$.*

(a) *Given that 4 wings were left at the end of the meal, what must be the relationship between α and A?*

Answer: Since the $MWTP$ goes to zero at the 16th wing, it must be that $A - 16\alpha = 0$ or $\alpha = A/16$.

(b) *Suppose $A = 8/3$. Was my brother right to want to order 20 instead of 10 wings?*

Answer: In order to answer this, we need to calculate the size of the shaded triangle in Graph 10.5 that depicts the total willingness to pay for 6 additional wings (beyond the first 10). When $A = 8/3$, it must be (given our answer to (a) above) that $\alpha = (8/3)/16 = 1/6$. This implies that the marginal willingness to pay of the 10th wing is $8/3-(1/6)10 = 1$. The triangle therefore has height of 1 — which means its area is $(1)(6)(1/2) = 3$. Thus, the value of the additional 6 wings is $3 — exactly the amount we paid for them. So both my brother and I were right — our consumer surplus did not increase or decrease from the decision to order 20 instead of 10 wings.

(c) *Suppose instead that $A = 2$. Does your answer change? What if $A = 4$?*

Answer: If $A < 8/3$, the relevant triangle will shrink below $3, and if $A > 8/3$, it will increase above $3. Thus, in the former case my brother was wrong, in the latter he was right. (To be more precise, if $A = 2$, then we know $\alpha = 2/16 = 1/8$. Thus, the marginal value of the 10th wing is $2-(1/8)10 = 3/4$. This implies that the area of the triangle is $(3/4)(6)(1/2) = 9/4$ — i.e. the value of the additional wings is only $2.25. If $A = 4$, on the other hand, $\alpha = 4/16 = 1/4$. This implies that the marginal value of the 10th wing is $4 - (1/4)10 = 3/2$ — which in turn implies that the area of the relevant triangle is $(3/2)(6)(1/2) = 9/2$. In that case, the value of the additional wings is therefore $4.50.)

(d) *If our tastes were Cobb-Douglas, could it ever be the case that we leave wings on the table?*

Answer: Under Cobb-Douglas tastes, the MRS never goes to zero no matter how much of a good one consumes. Thus, the $MWTP$ never falls to zero — which implies that, when wings are sitting on the table for anyone at the table to consume, they will always be consumed. So no wings should be left on the table under Cobb-Douglas tastes. (Caveat: One could extend this answer a bit further and make Cobb-Douglas tastes consistent with some wings being left on the table if one introduced an additional constraint. Suppose, for instance, that people at the table observe a per-meal calorie constraint — in that case, the constraint could cause some wings to be left.)

Exercise 10.7: *To Trade or Not to Trade Pizza Coupons*

Everyday Application: *To Trade or Not to Trade Pizza Coupons: Exploring the Difference between Willingness to Pay and Willingness to Accept: Suppose you and I are identical in every way — same exogenous income, same tastes over pizza and "other goods". The only difference between us is that I have a coupon that allows the owner of the coupon to buy as much pizza as he/she wants at 50% off.*

A: *Now suppose you approach me to see if there was any way we could make a deal under which I would sell you my coupon. Below you will explore under what conditions such a deal is possible.*

(a) *On a graph with pizza on the horizontal axis and "other goods" on the vertical, illustrate (as a vertical distance) the most you are willing to pay me for my coupon. Call this amount P.*

Answer: This is illustrated in the top graph in panel (a) of Graph 10.6 (next page). The steeper budget is your (without-coupon) budget, while the shallower budget is my (with coupon) budget that has half the price for pizza. Without a coupon, your optimal bundle is A — and you reach utility level u^A. Getting the coupon means getting the shallower slope for yourself — but buying it means that you are giving up income. In deciding the most you are willing to pay for a coupon, you therefore have to decide how much you are willing to shift the shallower budget in — and the most you are willing to shift it is an amount that will get you the same utility you can get without the coupon. Thus, the most you are willing to shift the coupon budget in is the amount that creates the tangency at B with u^A and the dashed budget (that has the same slope as the with-coupon budget. The vertical distance between my (with-coupon) budget and the dashed budget is the most you are willing to pay me for the coupon — a distance that can be measured anywhere between the two parallel lines. It is indicated as distance P in the graph.

(b) *On a separate but similar graph, illustrate (as a vertical distance) the least I would be willing to accept in cash to give up my coupon. Call this amount R.*

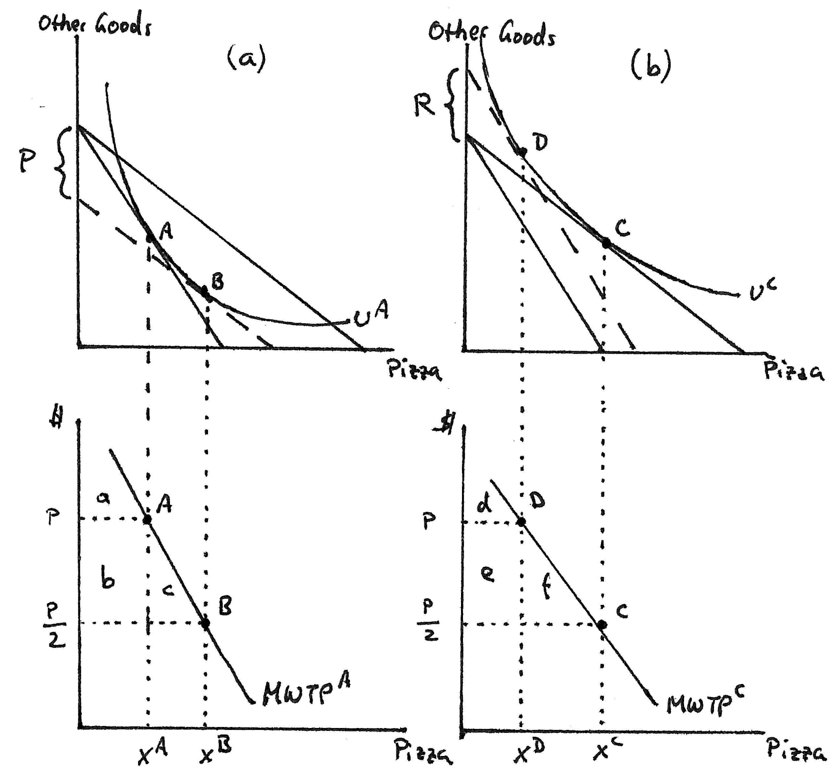
Answer: The top graph in panel (b) illustrates this. By just using the coupon, I will optimize at bundle C — and will reach the indifference curve u^C. Since I can get to that utility level without selling you the coupon, I will not be willing to make a deal that gets me less utility. By selling you the coupon, I will face a steeper budget but I will have received cash from you — i.e. I will face a budget with the steeper (no-coupon) slope but further out than your initial no-coupon budget. The least I am willing to accept for the coupon is an amount that will make me just as well off as I am with the coupon. I can determine that amount by taking your initial budget and shifting it out until it is tangent to my original optimal indifference curve u^C — which would land me at bundle D in the graph. The amount you have to give me in cash to get me to D is the vertical distance between your original (no-coupon) budget and the dashed budget in the graph. That distance, labeled R, is the least I am willing to accept for the coupon.

(c) *Below each of the graphs you have drawn in (a) and (b), illustrate the same amounts P and R (as areas) along the appropriate marginal willingness to pay curves.*

Answer: In the lower graph of panel (a), the marginal willingness to pay curve is derived from the indifference curve u^A. In the absence of a coupon, you will buy x^A in pizza at the no-coupon price p. This gives you consumer surplus of a. If you end up buying the coupon from me at the maximum price you are willing to pay (P), you will buy x^B in pizza and attain consumer surplus of $a+b+c$. Since A and B lie on the same indifference curve, you are equally happy attaining consumer surplus a without having paid me anything for the coupon or consumer surplus $a+b+c$ after paying me P for the coupon. In order for you to be truly indifferent between these two options, it must therefore be the case that $P=b+c$.

In the lower graph of panel (b), the marginal willingness to pay curve is derived from my indifference curve u^C. In the absence of selling my coupon, I buy x^C pizza — and get consumer surplus of $d+e+f$. If I sell the coupon at the lowest price R that I am willing to accept, I end up buying x^D pizza and get consumer surplus of just d. Since I am equally happy in both cases, it must be that I am indifferent between getting consumer surplus of $d+e+f$ without receiving any cash from you or getting consumer surplus d and getting R in cash. Thus, $R=e+f$.

(d) *Is P larger or smaller than R? What does your answer depend on? (*Hint*: By overlaying your lower graphs that illustrate P and R as areas along marginal willingness to pay curves, you*



Graph 10.6: Trading Pizza Coupons

should be able to tell whether one is bigger than the other or whether they are the same size depending on what kind of good pizza is.)

Answer: Asking if P is larger or smaller than R is then the same as asking if $b+c$ is larger or smaller than $e+f$. Suppose first that pizza is a quasilinear good for us. Then if I transferred the indifference curve u^C onto the top graph of panel (a), the tangency C would lie vertically above B — because a move from the dashed budget in panel (a) to my original (with-coupon) budget is simply an increase in income without a price change. Such an increase in income would not change consumption of pizza when pizza is quasilinear. Similarly, if we transferred the indifference curve u^A onto panel (b), the tangency at A would lie vertically below D. This is because the move from the dashed budget in panel (b) to your (no-coupon) budget is a simple decrease in income without a price change — which causes to change in consumption of pizza when pizza is quasilinear. This implies that, in the lower graphs, A lies at exactly the same place as D and B lies at exactly the same place as C. Put differently, when pizza is a quasilinear good, $MWTP^A$ lies exactly on top of $MWTP^C$ — which implies $b+c=e+f$ or $P=R$. The most you are willing to pay me for the coupon is then exactly equal to the least I am willing to accept.

Now suppose that pizza is an inferior good. Then the same logic we just went through implies that D will lie to the left of A and C will lie to the left of B — which implies that $b + c > e + f$ or $P > R$. Thus, when pizza is an inferior good, the most you are willing to pay is greater than the least I am willing to accept for the coupon. If, on the other hand, pizza is a normal good, then the same logic implies that D lies to the right of A and C lies to the right of B — which further implies that $b + c < e + f$ of $P < R$. Thus, when pizza is a normal good for us, then the most you are willing to pay is less than the least I am willing to accept for the coupon.

(e) True or False*: You and I will be able to make a deal so long as pizza is not a normal good. Explain your answer intuitively.*

Answer: This is true. We have just concluded that when pizza is an inferior good, you are willing to pay me more than the least I am willing to accept — so there is room for us to make a deal and both become better off. When pizza is quasilinear (i.e. borderline between normal and inferior), then the least I am willing to accept is exactly the most you are willing to pay — so in principle we can make a deal but neither one of us will be better or worse off for it. But when pizza is a normal good for us, the least I am willing to accept is more than them most you are willing to pay — so there is no way we will be able to strike a deal.

Intuitively, this makes sense in the following way: We began by saying that you and I are identical in every way — except there is one way in which we are not identical: I have a coupon and you do not. Thus, I am in essence richer than you are to begin with. If pizza is a normal good, then richer people will buy more pizza than poorer people — and so the coupon has more value to richer people because they would use it more. It is for this reason that we can't make a deal if pizza is normal for us. But if pizza is an inferior good, then being richer means I will want less pizza — and so I have less use for the coupon than you do. As a result, you will be willing to pay more than the least I am willing to accept. And if pizza is quasilinear, rich and poor buy the same amount of pizza — and thus make the same use of the coupon. Thus, if pizza is quasilinear, the coupon is worth the same to us.

B: *Suppose your and my tastes can be represented by the Cobb-Douglas utility function* $u(x_1, x_2) = x_1^{0.5} x_2^{0.5}$*, and suppose we both have income* $I = 100$*. Let pizza be denoted by* x_1 *and "other goods" by* x_2*, and let the price of pizza be denoted by* p*. (Since "other goods" are denominated in dollars, the price of* x_2 *is implicitly set to 1.)*

(a) *Calculate our demand functions for pizza and other goods as a function of* p*.*

Answer: Solving the problem

$$\max_{x_1, x_2} x_1^{1/2} x_2^{1/2} \text{ subject to } px_1 + x_2 = 100, \tag{10.21}$$

we get $x_1 = 50/p$ and $x_2 = 50$.

(b) *Calculate our compensated demand for pizza* (x_1) *and other goods* (x_2) *as a function of* p *(ignoring for now the existence of a coupon).*

Answer: Solving the problem

$$\min_{x_1, x_2} px_1 + x_2 \text{ subject to } u = x_1^{1/2} x_2^{1/2}, \tag{10.22}$$

we get $x_1 = u/(p^{1/2})$ and $x_2 = p^{1/2} u$.

(c) *Suppose* $p = 10$ *and the coupon reduces this price by half (to 5). Assume again that I have a coupon but you do not. How much utility do you and I get when we make optimal decisions?*

Answer: Using our (uncompensated) demand functions, we can calculate that your pizza consumption is $x_1 = 50/10 = 5$ while mine is $x_1 = 50/5 = 10$. This corresponds to x_A and x_D in Graph 10.6. Both of us consume 50 in other goods. Thus, your utility is $u(5,50) = 5^{1/2} 50^{1/2} \approx 15.81$ and my utility is $u(10,50) = 10^{1/2} 50^{1/2} \approx 22.36$.

(d) *How much pizza will you consume if you pay me the most you are willing to pay for the coupon? How much will I consume if you pay me the least I am willing to accept?*

Answer: If you pay me the most you are willing to pay, you will remain at the same utility level but will pay only half as much (i.e. \$5 instead of \$10). The compensated demand function therefore tells us that you will consume $x_1 = 15.81/(5^{1/2}) \approx 7.07$. If you pay me the least I am willing to accept, I will end up with the same utility as before but with a price that is twice as high (i.e. \$10 instead of \$5). Thus, the compensated demand function tells me that $x_1 = 22.36/(10^{1/2}) \approx 7.07$. In terms of the graphs in Graph 10.6, this implies that $x_B = 7.07 = x_D$.

(e) *Calculate the expenditure function for me and you.*

Answer: To get the expenditure function, we substitute the compensated demands back into the objective function of the minimization problem — i.e. we substitute $x_1 = u/(p^{1/2})$ and $x_2 = p^{1/2}u$ into $px_1 + x_2$ to get

$$E(p,u) = p\left(\frac{u}{p^{1/2}}\right) + p^{1/2}u = 2p^{1/2}u. \tag{10.23}$$

(f) *Using your answers so far, determine R — the least I am willing to accept to give up my coupon. Then determine P — the most you are willing to pay to get a coupon.(*Hint*: Use your graphs from A(a) to determine the appropriate values to plug into the expenditure function to determine how much income I would have to have to give up my coupon. Once you have done this, you can subtract my actual income $I = 100$ to determine how much you have to give me to be willing to let go of the coupon. Then do the analogous to determine how much you'd be willing to pay, this time using your graph from A(b).)*

Answer: The budget required for me to be just as happy without the coupon (i.e. when $p = 10$) is the expenditure necessary for me to reach utility level 22.36 (which is u^C in our graph) at $p = 10$ — i.e. $E(10, 22.36) = 2(10^{1/2})22.36 \approx 141.42$. Since I started out with an income of \$100, this implies that you would have to give me approximately \$41.42 for the coupon in order for me to be just as happy; i.e. $R = 41.42$. The budget required for you to be just as happy with the coupon as you were without is the expenditure necessary to get you to utility level 15.81 (which is u^A in our graph) at the with-coupon price of 5 — i.e. $E(5, 15.81) = 2(5^{1/2})15.81 \approx 70.71$. Since you started with an income of \$100, this means you would be willing to pay me as much as $100 - 70.71 = 29.29$ to get the coupon; i.e. $P = 29.29$.

(g) *Are we able to make a deal under which I sell you my coupon? Make sense of this given what you found intuitively in part A and given what you know about Cobb-Douglas tastes.*

Answer: No, we are not able to make a deal since the most you are willing to pay me (\$29.29) is less than the least I am willing to accept (\$41.42). This is consistent with what we concluded in part A where we said that we would not be able to strike a deal if pizza is a normal good for us. Cobb-Douglas tastes are tastes over normal goods — so under the tastes represented by the utility function we have been working with, pizza is in fact a normal good.

(h) *Now suppose our tastes could instead be represented by the utility function $u(x_1, x_2) = 50\ln x_1 + x_2$. Using steps similar to what you have just done, calculate again the least I am willing to accept and the most you are willing to pay for the coupon. Explain the intuition behind your answer given what you know about quasilinear tastes.*

Answer: Solving the problem

$$\max_{x_1,x_2} 50\ln x_1 + x_2 \text{ subject to } px_1 + x_2 = 100, \tag{10.24}$$

we get the (uncompensated) demands $x_1 = 50/p$ and $x_2 = 50$. Thus, both you and I consume 50 in other goods, but I consume 10 pizzas while you only consume 5 because I face a with-coupon price of \$5 per pizza while you face a without-coupon price of \$10 per pizza.

Plugging $(x_1, x_2) = (10, 50)$ into the utility function, we get my utility (equivalent to u^C in the graph) of 165.13. Plugging $(x_1, x_2) = (5, 50)$ into the utility function for you, we get your utility (equivalent to u^A in our graph) as 130.47.

Solving the minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = 50\ln x_1 + x_2, \tag{10.25}$$

we can derive the compensated demands $x_1 = 50/p$ and $x_2 = u - 50\ln(50/p)$. (Note that the compensated and uncompensated demands for the quasilinear good x_1 are the same — which makes sense since there are no income effects to make the two demands different.)

Next, we can find the expenditure function by just plugging the compensated demands into the objective function of the minimization problem to get

$$E(p,u) = p\left(\frac{50}{p}\right) + u - 50\ln\left(\frac{50}{p}\right) = u + 50\left(1 - \ln\left(\frac{50}{p}\right)\right). \tag{10.26}$$

To determine the expenditure necessary for me to get to my current utility level in the absence of the coupon (i.e. when $p = 10$ instead of $p = 5$), we calculate $E(10, 165.13) \approx 134.66$. Since I start with an income of \$100, that means the least I am willing to accept for the coupon is $R = 134.66 - 100 = \$34.66$. To calculate the expenditure necessary to get you to your current utility level in the presence of a coupon (i.e. when $p = 5$ instead of $p = 10$), we calculate $E(5, 130.47) \approx 65.34$. Since you also start with an income of \$100, this means that the most you are willing to pay for the coupon is $P = 100 - 65.34 = \$34.66$. Thus $P = R$ as we concluded it has to be when pizza is a quasilinear good.

(i) *Can you demonstrate, using the compensated demand functions you calculated for the two types of tastes, that the values for P and R are in fact areas under these functions (as you described in your answer to A(c)? (*Note*: This part requires you to use integral calculus.)*

Answer: The compensated demand function for pizza is $x_1 = 50/p$ (as calculated in the previous part). The area in the graph is the integral under this function between the with-coupon and the without-coupon prices. (Note: In our graphs, we are graphing the inverse of the compensated demand functions — which is why the area appears as an area to the left of the curve rather than an area under the curve.) Thus, the least I am willing to accept (R) is

$$R = \int_5^{10} \frac{50}{p}\,dp = 50\ln p|_5^{10} = 50\ln 10 - 50\ln 5 = 34.66. \tag{10.27}$$

Since, because of the quasilinearity of pizza, our compensated demand functions are the same (because u does not appear in the functions), the same holds for P. Thus, $P = R = 34.66$, exactly as we concluded above.

Exercise 10.10: *Pricing at Disneyland*

Business Application: *Pricing at Disneyland: In the 1970s, Disneyland charged an entrance fee to get into the park and then required customers to separately buy tickets for each ride once they were in the park. In the 1980s, Disneyland switched to a different pricing system that continues to this day. Now, customers simply pay an entrance fee and then all rides in the park are free.*

A: *Suppose you own an amusement park with many rides (and assume, for the sake of simplicity, that all rides cost the same to operate.) Suppose further that the maximum number of rides a customer can take on any given day (given how long rides take and how long the average wait times are) is 25. Your typical vacationing customer has some exogenous daily vacation budget I to allocate between rides at your park and other forms of entertainment (that are, for purposes of this problem) bought from vendors other than you. Finally, suppose tastes are quasilinear in amusement park rides.*

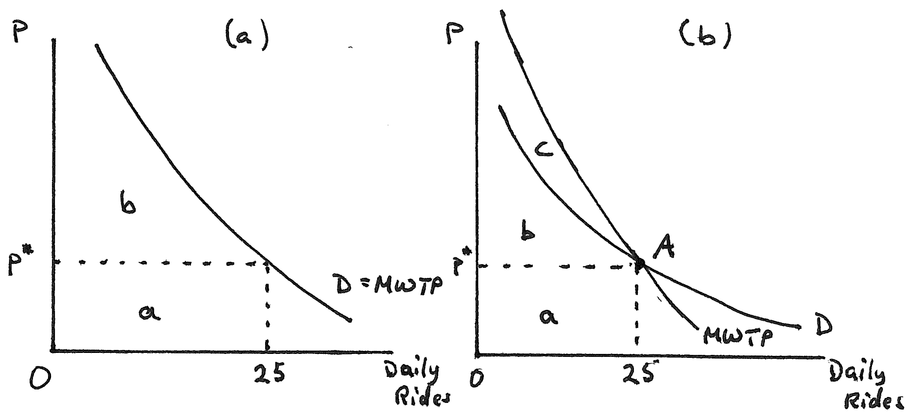
(a) *Draw a demand curve for rides in your park. Suppose you charge no entrance fee and only charge your customers per ride. Indicate the maximum price per ride you could charge while insuring that your consumer will in fact spend all her day riding rides (i.e. ride 25 times).*

Answer: This is illustrated in panel (a) of Graph 10.7 (next page) where p^* indicates the maximum price per ride such that the consumer will choose the maximum number of 25 rides per day.

(b) *On your graph, indicate the total amount that the consumer will spend.*

Answer: This is indicated in panel (a) as area a — the price p^* times the quantity 25.

(c) *Now suppose that you decide you want to keep the price per ride you have been using but you'd also like to charge a separate entrance fee to the park. What is the most you can charge your customer?*



Graph 10.7: Pricing at Disneyland

Answer: Since tastes are quasilinear in x_1, the demand curve is also the $MWTP$ curve. We can therefore measure the consumer's total willingness to pay for rides as the area underneath the demand curve up to 25. Since the consumer is already spending $\$25p^* = a$, the remaining amount we could charge as an entrance fee is equal to the area b.

(d) *Suppose you decide that it is just too much trouble to collect fees for each ride — so you eliminate the price per ride and switch to a system where you only charge an entrance fee to the park. How high an entrance fee can you charge?*

Answer: If we lower the price per ride to zero (and there is still a maximum of 25 rides per day that the consumer has time for), we can charge an entrance fee equal to the entire area below the $MWTP$ curve — i.e. the area $a+b$.

(e) *How would your analysis change if x_1, amusement part rides, is a normal good rather than being quasilinear?*

Answer: In that case, the $MWTP$ curve is no longer the same as the demand curve. If x_1 is a normal good, the $MWTP$ that crosses the demand curve at A in panel (b) of Graph 10.7 is steeper than the demand curve — because the income effect that is included in the demand curve points in the same direction as the substitution effect that is included in both curves. Thus, the total willingness to pay is $a+b+c$ — which means we can charge an entrance fee of $b+c$ if we continue to charge a per-ride fee of p^* and we can charge an entrance fee of $a+b+c$ if we charge no price per ride.

B: *Consider a consumer on vacation who visits your amusement park for the day. Suppose her tastes can be summarized by the utility function $u(x_1,x_2) = 10x_1^{0.5} + x_2$ where x_1 represents daily rides in the amusement park and x_2 represents dollars of other entertainment spending. Suppose further that her exogenous daily budget for entertainment is \$100.*

(a) *Derive the uncompensated and compensated demand functions for x_1 and x_2.*

Answer: To solve for uncompensated demands, we solve

$$\max_{x_1,x_2} 10x_1^{0.5} + x_2 \text{ subject to } px_1 + x_2 = 100, \quad (10.28)$$

which gives us

$$x_1 = \frac{25}{p^2} \text{ and } x_2 = 100 - \frac{25}{p}. \quad (10.29)$$

To solve for compensated demands, we solve

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = 10x_1^{0.5} + x_2, \tag{10.30}$$

which gives us

$$x_1 = \frac{25}{p^2} \text{ and } x_2 = u - \frac{50}{p}. \tag{10.31}$$

(b) *Suppose again there is only enough time for a customer to ride 25 rides a day in your amusement park and suppose that congestion and wear-and-tear on equipment in the park is not a problem. Suppose then that you'd like your customer to ride as much as possible so he can spread the word on how great your rides are. What price will you set per ride?*

Answer: You would like to set a price such that your consumer demands exactly 25 rides per day. Your consumer's demand function is $x_1 = 25/(p^2)$. Setting $x_1 = 25$, we can then solve for the price per ride of $p = 1$.

(c) *How much utility will your consumer attain under your pricing?*

Answer: Your consumer will consume $x_1 = 25$ and $x_2 = 100 - (25/1) = 75$. Plugging this bundle into the utility function, we get $u(25,75) = 10(25^{0.5}) + 75 = 125$.

(d) *Suppose you can also charge an entrance fee to your park — in addition to charging the price per ride you calculated above. How high an entrance fee would you charge? (*Hint*: You should be evaluating an integral, which draws on some of the material from the appendix.)*

Answer: The total willingness to pay is an area under the appropriate compensated demand curve — i.e. the compensated demand curve that corresponds to the indifference curve at the bundle at which the consumer consumes. Since tastes are quasilinear in x_1, the compensated demand function is identical to the uncompensated demand function for x_1. Thus, we simply need to evaluate the area under the demand curve up to the quantity $x_1 = 25$ that is consumed. The demand curve, with p on the vertical axis, is the inverse of the demand function $x_1 = 25/(p^2)$ — i.e. $p = 5/(x_1^{0.5})$. We therefore need to evaluate

$$\int_0^{25} \frac{25}{x_1^{0.5}}\, dx_1 = 10x_1^{0.5}|_0^{25} = 10(25^{0.5}) - 10(0^{0.5}) = 50. \tag{10.32}$$

This means that the consumer's total willingness to pay for the 25 rides is \$50. By charging \$1 per ride for 25 rides, we are charging the consumer \$25. Thus, we can charge an entrance fee of an additional \$25.

(e) *Now suppose you decide to make all rides free (knowing that the most rides the consumer can squeeze into a day is 25) and you simply charge an entrance fee to your park. How high an entrance fee will you now charge to your park? (Note: This part is not computationally difficult — it is designated with ** only because you have to use information from the previous part.)*

Answer: If you no longer charge per ride, you are in essence charging the consumer $p = 0$. Since the consumer chooses to take the maximum 25 rides even at a price of $p = 1$, she will still take all 25 rides. But now she incurs no cost per ride once she is in the park — and we calculated above that she values 25 rides at \$50. Thus, we would charge her an entrance free of \$50.

(f) *How does your analysis change if the consumer's tastes instead were given by $u(x_1,x_2) = (3^{-0.5})x_1^{0.5} + x_2^{0.5}$?*

Answer: When x_1 was quasilinear, the compensated and uncompensated demand curves were the same (because there are no income effects). Now, the two will differ. In particular, solving the maximization problem, we get uncompensated demands (assuming again a daily budget of \$100)

$$x_1 = \frac{100}{p(1+3p)} \text{ and } x_2 = \frac{300p}{1+3p}. \tag{10.33}$$

Solving the expenditure minimization problem, we get compensated demands

$$x_1 = 3\left(\frac{u}{1+3p}\right)^2 \text{ and } x_2 = \left(\frac{3pu}{1+3p}\right)^2. \tag{10.34}$$

In order for the consumer to demand exactly 25 rides at a per-ride price p, we simply set the uncompensated demand for x_1 equal to 25 and solve for p; i.e. we solve

$$25 = \frac{100}{p(1+3p)} \tag{10.35}$$

which gives us $p = 1$, identical to our result from before.[1] At $p = 1$, the consumer will consume the bundle $(x_1, x_2) = (25, 75)$ which gives utility $u = (3^{-0.5})(25^{0.5}) + 75^{0.5} \approx 11.547$.

To calculate the consumer's total willingness to pay, we need to calculate the appropriate area under the *compensated* demand curve that corresponds to this utility level — which is the inverse of the compensated demand function

$$x_1 = 3\left(\frac{11.547}{1+3p}\right)^2; \tag{10.36}$$

i.e. it is this function solved for p —

$$p = \frac{20}{3x_1^{0.5}} + \frac{1}{3}. \tag{10.37}$$

Taking the integral of this and evaluating it from $x_1 = 0$ to $x_1 = 25$, we get the total willingness to pay for 25 rides as

$$\int_0^{25} \left(\frac{20}{3x_1^{0.5}} - \frac{1}{3}\right) dx_1 = \frac{40x_1^{0.5} - x_1}{3}\Big|_0^{25} = \frac{175}{3} \approx 58.33. \tag{10.38}$$

Under the price per ride of \$1 per ride, the consumer is already paying \$25. If you were to charge an entrance fee plus a price of \$1 per ride, you would therefore be able to charge a fee of \$58.33-\$25=\$33.33. If you abandoned the policy of charging per ride and only charged an entrance fee, you would be able to set the entrance fee at \$58.33.

Exercise 10.13: *Price Subsidies*

Policy Application: *Price Subsidies*: *Suppose the government decides to subsidize (rather than tax) consumption of grits.*

A: *Consider a consumer that consumes boxes of grits and "other goods".*

(a) *Begin by drawing a budget constraint (assuming some exogenous income) with grits on the horizontal axis and "other consumption" on the vertical. Then illustrate a new budget constraint with the subsidy — reflecting that each box of grits now costs the consumer less than it did before.*

Answer: This is illustrated in the top graph of Graph 10.8 (next page), with $(p - s)$ indicating the price with the subsidy and p indicating the price without.

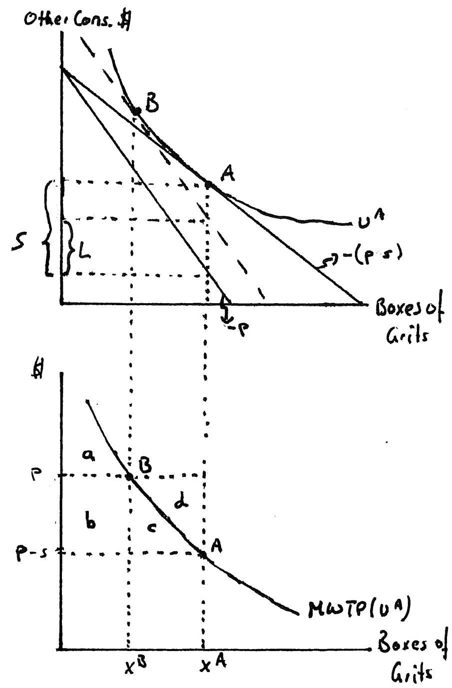
(b) *Illustrate the optimal consumption of grits with an indifference curve tangent to the after-subsidy budget. Then illustrate in your graph the amount that the government spends on the subsidy for you. Call this amount S.*

Answer: The optimal consumption bundle is illustrated as bundle A. The vertical intercept of that bundle indicates how much in other consumption the consumer is able to afford given that grits are subsidized. Had they not been subsidized, a much lower amount (read off the no-subsidy budget) would be available for other consumption. The difference is S — the amount the government paid for this consumer under the price subsidy policy.

(c) *Next, illustrate how much the government could have given you in a lump sum cash payment instead and made you just as happy as you are under the subsidy policy. Call this amount L.*

Answer: The government could have chosen not to alter the price of grits (and thus not alter the slope of the no-subsidy budget line) but instead simply shift that budget out in a

[1] We have to solve the equation $3p^2 + p - 4 = 0$ using the quadratic formula — which gives us two solutions: $p = 1$ and $p = -4/3$. Since the latter is negative, it is not economically meaningful.



Graph 10.8: Subsidizing Grits

parallel way by giving a cash subsidy. The amount in cash the government could have given to make the consumer just as happy as she is under the price subsidy is then an amount that creates the dashed budget which is tangent to the post-subsidy indifference curve u^A. This tangency occurs at bundle B, and the cost of this cash subsidy is simply the vertical difference between the dashed budget and the parallel no-subsidy budget. That distance can be measured anywhere (since the lines are parallel) and is indicated as the distance L.

(d) *Which is bigger — S or L?*

Answer: S is bigger than L because of the substitution effect from A to B. You don't have to give someone as much in unrestricted cash as you would have to spend in a subsidy that is restricted to the purchase of one good.

(e) *On a graph below the one you have drawn, illustrate the relevant $MWTP$ curve and show where S and L can be found on that graph.*

Answer: The graph below the top graph derives the $MWTP$ or compensated demand curve that corresponds to utility level u^A. Under the price subsidy, the consumer consumes at A — which gives consumer surplus of $a+b+c$. The government is paying the difference between p and $(p-s)$ for each of the x^A boxes of grits the consumer buys — which means that the cost of the price subsidy is $S = b+c+d$.

Under the cash subsidy, the consumer faces the higher price p (rather than $(p-s)$) and buys x^B rather than x^A assuming she receives the cash subsidy L. This leaves her with consumer surplus of a in the grits market — but she is equally happy since both A and B lie on the same indifference curve. The only way she can be equally happy is if the cash subsidy was enough to make up for the loss in consumer surplus in the grits market — i.e. $L = b + c$.

(f) *What would your tastes have to be like in order for S to be equal to L.*

Answer: The substitution effect that creates the difference between S and L would have to disappear — which happens only if there is a sharp kink in the indifference curve at A (such as if grits and other goods are perfect complements). In that case, $A = B$ and $L = S$. In the lower graph, this implies that B lies directly above A — leading to a perfectly vertical $MWTP$ curve and the disappearance of the area d.

(g) *True or False: For almost all tastes, price subsidies are inefficient.*

Answer: This is true — so long as there aren't sharp kinks in just the right places of indifference curves — i.e. so long as goods are somewhat substitutable at the margin, $S > L$ which leaves the difference as a deadweight loss. If the substitutability goes away, so does the deadweight loss triangle d as the $MWTP$ curve becomes vertical.

B: *Suppose the consumer's tastes are Cobb-Douglas and take the form $u(x_1, x_2) = x_1^{\alpha} x_2^{(1-\alpha)}$ where x_1 is boxes of grits and x_2 is a composite good with price normalized to 1. The consumer's exogenous income is I.*

(a) *Suppose the government price subsidy lowers the price of grits from p to $(p - s)$. How much S will the government have to pay to fund this price subsidy for this consumer?*

Answer: We need to solve the problem

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{(1-\alpha)} \text{ subject to } px_1 + x_2. \tag{10.39}$$

This solves to

$$x_1 = \frac{\alpha I}{p} \text{ and } x_2 = (1-\alpha)I. \tag{10.40}$$

When the government lowers the price to $(p - s)$, demand is $x_1 = \alpha I/(p - s)$. For each box of grits, the consumer pays $(p - s)$ while the government pays s. Thus, the government's expense is

$$S = \frac{s\alpha I}{p - s}. \tag{10.41}$$

(b) *How much utility does the consumer attain under this price subsidy?*

Answer: Under the price subsidy, the consumer chooses the bundle $(x_1, x_2) = (\alpha I/(p - 2), (1-\alpha)I)$. Substituting this into the utility function, we get the indirect utility function

$$V(p, s) = \left(\frac{\alpha I}{p - s}\right)^{\alpha} ((1-\alpha)I)^{(1-\alpha)} = \frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p - s)^{\alpha}} I. \tag{10.42}$$

(c) *How much L would the government have had to pay this consumer in cash to make the consumer equally happy as she is under the price subsidy?*

Answer: Given that we know how much utility the consumer gets under the price subsidy, we now have to ask what expenditure (in cash) would have been necessary to get to the same utility level at the non-subsidized price p. We can derive the expenditure function by solving the expenditure minimization problem

$$\min_{x_1, x_2} px_1 + x_2 \text{ subject to } u = x_1^{\alpha} x_2^{(1-\alpha)}, \tag{10.43}$$

and then plug the compensated demands into the objective $px_1 + x_2$, or we can simply invert the indirect utility function. Either way, we get

$$E(p, u) = \frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}. \tag{10.44}$$

We are interested in knowing the expenditure necessary at p to get to utility level $V(p, s)$ from equation (10.42); i.e. we are interested in

$$E(p,V(p,s)) = \left(\frac{p^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)\left(\frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{(p-s)^{\alpha}}I\right) = \left(\frac{p}{p-s}\right)^{\alpha} I. \tag{10.45}$$

This is the total expenditure necessary to get to the price-subsidy utility level $V(p,s)$. Since the consumer starts with an income I, the amount of cash L the consumer would have to get to be equally happy as under the price subsidy would therefore be

$$L = E(p,V(p,s)) - I = \left[\left(\frac{p}{p-s}\right)^{\alpha} - 1\right] I. \tag{10.46}$$

(d) *What is the deadweight loss from the price subsidy?*

Answer: The deadweight loss is then just the difference between what the government spends under the price subsidy (S) and what the government could have spent in a lump sum way (L) to make the consumer just as well off. This is

$$DWL = S - L = \left[1 + \frac{s\alpha}{p-s} - \left(\frac{p}{p-s}\right)^{\alpha}\right] I. \tag{10.47}$$

(e) *Suppose* $I = 1000$, $p = 2$, $s = 1$ *and* $\alpha = 0.5$. *How much grits does the consumer buy before any subsidy, under the price subsidy and under the utility-equivalent cash subsidy? What is the deadweight loss from the price subsidy?*

Answer: We have calculated that the demand for grits is $x_1 = \alpha I/p$. Thus, when the price is unsubsidized originally, the consumer buys $x_1 = 0.5(1000)/2 = 250$. The price subsidy lowers the effective price for the consumer to 1 — which implies the new quantity demanded is $x_1 = 0.5(1000)/1 = 500$. To calculate the equivalent cash subsidy, we can use equation (10.46) to get

$$L = \left[\left(\frac{2}{2-1}\right)^{0.5} - 1\right](1000) \approx 414.21. \tag{10.48}$$

The consumer's income under the cash subsidy would therefore rise to $1,414.21 but the price would remain at $p = 2$. The consumer's demand would therefore be $x_1 = 0.5(1414.21)/2$ which is approximately 354.

Finally, the deadweight loss is simply $(S - L)$. The cost of the price subsidy, given that the consumer will demand 500 units of x_1 and the cost of the subsidy is $1 per unit, is $500. The deadweight loss is therefore 500-414.21=$85.79. You can also get this by simply plugging the relevant values into the DWL equation we calculated in equation (10.47); i.e.

$$DWL = \left[1 + \frac{1(0.5)}{2-1} - \left(\frac{2}{2-1}\right)^{0.5}\right](1000) \approx 85.79. \tag{10.49}$$

(f) *Continue with the values from he previous part. Can you calculate the compensated demand curve you illustrated in A(e) and verify that the area you identified as the deadweight loss is equal to what you have calculated? (*Hint*: You need to take an integral and use some of the material from the appendix to answer this.)*

Answer: The compensated demand curve arises from the expenditure minimization problem

$$\min_{x_1,x_2} px_1 + x_2 \text{ subject to } u = x_1^{0.5}x_2^{0.5}. \tag{10.50}$$

From this, we get compensated demands

$$x_1 = \frac{u}{p^{0.5}} \text{ and } x_2 = p^{0.5}u. \tag{10.51}$$

Using the indirect utility function $V(p,s)$ from equation (10.42), we can determine the utility the consumer gets under the price subsidy as

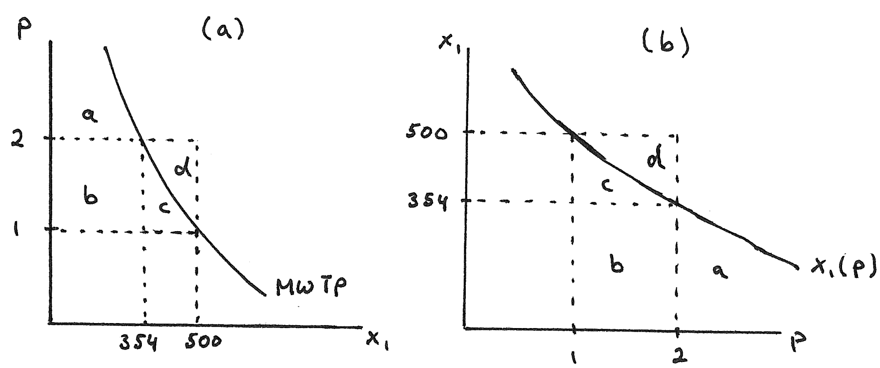
$$V(2,1) = \left(\frac{0.5^{0.5}0.5^{0.5}}{(2-1)^{0.5}}\right)(1000) = 500. \tag{10.52}$$

The appropriate compensated (or $MWTP$) curve is then $x_1 = 500/(p^{0.5})$. The inverse of this function is sketched in panel (a) of Graph 10.9 — it is similar to the lower graph in Graph 10.8 where we indicated that $S = b + c + d$, $L = b + c$ and $DWL = d$. Panel (b) of Graph 10.9 then simply inverts panel (a), placing x_1 on the vertical (rather than the horizontal) and p on the horizontal (rather than the vertical) axes.

The area $L = c + b$ is then simply the integral under the function $x_1 = 500/(p^{0.5})$ evaluated from $p = 1$ to $p = 2$. This is

$$\int_1^2 \frac{500}{p^{0.5}}\,dp = 2(500)p^{0.5}|_1^2 = 1000(2^{0.5} - 1) \approx 414.21. \tag{10.53}$$

This is exactly what we calculated for L in the previous part. The area $S = b + c + d$ is simply 500. Thus, the deadweight loss is $DWL = 500 - 414.21 = 85.79$, again exactly as we calculated before.



Graph 10.9: Subsidizing Grits: Part 2

Conclusion: Potentially Helpful Reminders

1. The regular (or uncompensated) demand curve is always the one you want to use if you are trying to predict what consumers will actually do as a result of a price change (regardless of what causes that price change).

2. The compensated demand (or marginal willingness to pay) curve is always the one you want to use when assessing changes in consumer surplus that result from price changes.

3. Since deadweight loss is a loss of consumer surplus, it is always measured on the compensated demand curve.

4. If the underlying good is quasilinear, the compensated and uncompensated demand curves are the same curves — which implies that this is the one case where you can measure consumer surplus changes along regular (uncompensated) demand curves.

5. Since marginal willingness to pay curves arise from substitution effects, they will be steep for small substitution effects and shallow for large substitution effects.

6. For students who do the B-part of the chapter, note the multiple ways we have found to calculate the various functions in the duality picture. Note further that this allows us multiple ways of calculating things like changes in consumer surplus or deadweight losses. In particular, we can either use the expenditure function — or we can take integrals on the compensated demand curve. A number of the end-of-chapter exercises illustrate the equivalence of these two methods.

7. Again for students who do the B-part of the chapter: Be sure to understand the duality picture well. You might be given one of the functions in the picture and then asked to tackle a problem — which is hard unless you know how to get from one part of the duality picture to another. For instance, if you already have the indirect utility function and you need the expenditure function, it is much easier to simply invert the indirect utility function rather than setting up the expenditure minimization problem and solving it. But be sure not to memorize but rather try to understand why the functions are related to one another as the picture shows.