CHAPTER

11

One Input and One Output: A Simple Model of Producers

In this chapter, we begin to look at the theory of the firm as a price taker. We do this in the simplest possible setting — where producers use a single input to produce a single output — because it is in this setting that we can develop the basics of profit maximization. As will become clearer in upcoming chapters, this one-input/one-output model can be thought of as a *short run* model of producer choice because we can think of inputs (other than the one input in the model) as being fixed in the short run. So, for instance, when we say "labor is the only input", one interpretation of this statement is that it really means "labor is the only input that can be varied in the short run" because other inputs like factory size can only be changed in the long run.

Chapter Highlights

The main points of the chapter are:

1. A **production plan** specifies a bundle of inputs and outputs just as a consumption bundle in consumer theory represents a bundle of different goods. Just as consumers choose consumption bundles that maximize utility, **producers choose production plans that maximize profit.** Both try "to do the best they can given their circumstances," but — unlike in the consumer case — what is "best" has a more concrete meaning because we can quantify profit as the difference between economic revenues and costs.

2. As a result, "indifference curves" for producers — or **isoprofit curves** — are not a matter of "tastes" but emerge from the fact that producers are indifferent between production plans that result in the same level of profit. Since the profit of any production plan depends on the input and output prices, this implies that these producer "indifference curves" emerge from prices. This is unlike the consumer case where prices have nothing to do with indifference curves.

3. The fundamental constraint the producers face is a **technological constraint** that limits which production plans are in fact technologically feasible. Unlike the consumer case where constraints are shaped by prices, the technological constraint — modeled through the **production frontier** or the **production function** — has nothing to do with prices.

4. Any profit maximizing production plan (that involves positive production) has the characteristic that the **marginal revenue product of the intput** equals the marginal cost of the input represented by the input's price. From this insight we can derive both the (short run) **output supply** curve (or function) as well as the (short run) **input demand** curve (or function), each sloping in the expected direction because of the **law of diminishing marginal product**.

5. There are **two ways to think of profit maximization**: (1) As a single problem in which we equate marginal revenue products with input prices, or (2) as a two-step problem in which the firm first derives its **cost curve** (or function) and then asks where the difference between revenues and costs is maximized. The latter allows us to state the profit maximizing condition as $MC = p$ for price taking firms.

6. Finally, **economic profit** is the difference between *economic* revenues and costs. This implies that, when a firm makes zero economic profit, it is doing as well in this industry as it could be in the next best alternative industry — it does not mean that "the firm isn't making money." Thus, it makes sense for firms to produce so long as economic profit is not negative.

Using the LiveGraphs

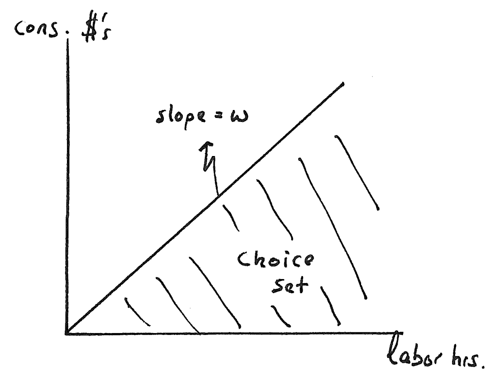
For an overview of what is contained on the LiveGraphs site for each of the chapters (from Chapter 2 through 29) and how you might utilize this resource, see pages 2-3 of Chapter 1 of this *Study Guide*. To access the LiveGraphs for Chapter 11, click the *Chapter 11* tab on the left side of the LiveGraphs web site.

While there are no additional *Exploring Relationships* modules for this chapter, you may find it useful to fast-forward a bit and play some of the *Exploring Relationships* modules for Chapter 12. In particular, these modules demonstrate what production frontiers look like when there are two inputs — and how we can think of the one-input production frontier in Chapter 11 as simply a slice of the more complete production technology, a slice that holds one input fixed.

11A Solutions to Within-Chapter Exercises for Part A

Exercise 11A.1 *Can you model a worker as a "producer of consumption" and interpret his choice set within the context of the single input, single output producer model?*

Answer: A worker uses the input "labor hours" to produce the output "dollars for consumption" by selling labor hours at the market wage w. This can be graphed with "labor hours" on the horizontal and "dollars of consumption" on the vertical, as in Graph 11.1 below. For every input hour, the worker gets w — thus implying a slope w of the production frontier.



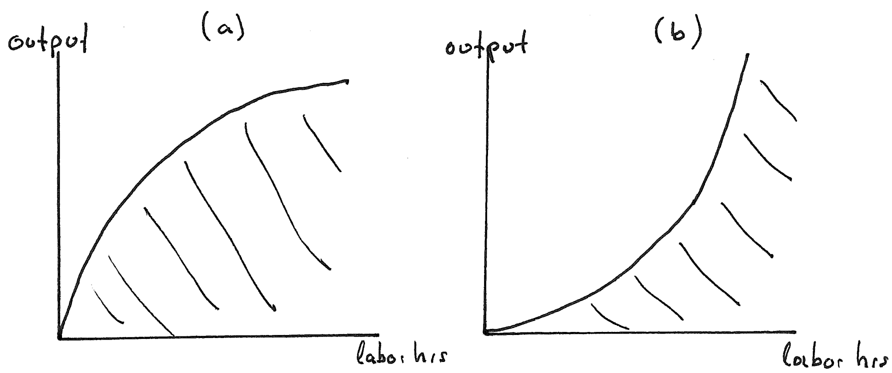
Graph 11.1: Worker as a "Producer of Consumption"

Exercise 11A.2 *Which of the producer choice sets in Graph 11.1 is non-convex? What makes it non-convex?*

Answer: The producer choice set in panel (b) is non-convex because you can choose two production plans (such as A' and C') such that the line connecting the two plans lies outside the set.

Exercise 11A.3 *Suppose my technology was such that each additional worker hour, beginning with the second one, is less productive than the previous. Would my producer choice set be convex? What if my technology was such that each additional worker hour, beginning with the second one, is more productive than the previous.*

Answer: The first of these is graphed in panel (a) of Graph 11.2 (next page) while the second is graphed in panel (b). Each worker becoming less productive than the previous leads to the increasingly shallow slope in (a) while each worker becoming more productive than the previous leads to the increasingly steep slope in (b). The first producer choice set is convex; the second is not.



Graph 11.2: Two different types of producer choice sets

Exercise 11A.4 *Under the production technology in Graph 11.1b, what is the approximate marginal benefit of hiring an additional labor hour when I already have 95 labor hours employed?*

Answer: We know from Graph 11.1(b) in the text that I can produce 390 cards per day with 90 labor hours and 400 cards per day with 100 labor hours. Thus, the additional 10 labor hours increase total output by 10 cards — which gives me a marginal benefit of workers in that range (including the 95th worker) of about 1 card per worker hour. This is also reflected in the marginal product graph (Graph 11.2(b) in the text) where the marginal product for the 95th worker hour is indicated as 1.

Exercise 11A.5 *Relate your answer from exercise 11A.4 to a point on the MP_ℓ curve plotted in Graph 11.2b.*

Answer: The MP_ℓ curve tells me that the 95th worker hour results in an increase in total output of 0.98 cards per day — which is the same as what we derived in the previous exercise from the production frontier. This is because the marginal product curve is just the slope of the production frontier, and between 90 and 100 labor hours, the slope of the production frontier is approximately 0.98.

Exercise 11A.6 *What would the MP_ℓ curves look like for the technologies described in within-chapter exercise 11A.3?*

Answer: For the first technology described in exercise 11A.3, the MP_ℓ curve would be downward sloping, and for the second it would be upward sloping.

Exercise 11A.7 True or False*: The Law of Diminishing Marginal Product implies that producer choice sets in single input models must be convex beginning at some input level.*

Answer: This is true. At the input level at which the law of diminishing marginal product sets in, the production frontier begins to become shallower and shallower — resulting in a choice set underneath that is convex from that point forward.

Exercise 11A.8 True or False*: If the Law of Diminishing Marginal Product did not hold in the dairy industry, I could feed the entire world milk from a single cow.* (Hint*: Think of the cow as a fixed input and feed for the cow as the variable input for which you consider the marginal product in terms of milk produced per day.)*

Answer: Take a given cow and begin feeding it. Each ounce of feed results in more milk per day from the cow than each previous ounce of feed. If the law of diminishing marginal product of feed does not at some point decrease, this means I can keep feeding the cow and I will get the cow to produce ever increasing amounts of milk (per day) from each additional ounce of feed. At some point, I will have given the cow so much feed that we will have produced enough milk for the world from that single cow. Obviously this is absurd, which illustrates why the assumption of diminishing marginal product of feed must hold.

Exercise 11A.9 *Without knowing what prices and wages are in the economy, can you tell by looking at a single isoprofit curve whether profits for production plans along this curve are positive or negative? What has to be true about an isoprofit curve in order for profit to be zero?*

Answer: The intercept of an isoprofit curve on the vertical axis is Profit divided by price. You can therefore tell whether profits along an isoprofit curve are positive by checking whether or not the isoprofit curve has positive intercept. If it has positive intercept, then profit for all the production plans along that isoprofit curve is positive (assuming price is positive); if it has negative intercept, then profit for all production plans along the curve is negative. If the isoprofit curve intersects at the origin, then profit is zero.

Exercise 11A.10 *What would have to be true in order for an isoprofit curve to have a negative slope?*

Answer: The slope of an isoprofit curve is w/p. In order for this slope to be negative, either wage or output price would therefore have to be negative.

Exercise 11A.11 *How would the blue isoprofit curve in Graph 11.3a change if the wage rises to \$30? What if instead the output price falls to \$2?*

Answer: Along the original blue isoprofit curve, wage is \$20 and output price is \$5 — resulting in a slope of $w/p = 20/5 = 4$. If the wage increases to \$30, the slope increases to $30/5 = 6$. Since the intercept is given by Profit/p (and w therefore does not enter the intercept term), the intercept remains at 40.

Now suppose that instead the price fell from \$5 to \$2. Originally, the intercept of the blue isoprofit is Profit/p=Profit/5=40 — which implies that Profit along the isoprofit curve is \$200. In order for profit to remain unchanged at the intercept when price falls to \$2, it would have to be the case that the new intercept is $200/2 = 100$. Thus, the intercept of the isoprofit would shift up from 40 to 100. The slope of the isoprofit is w/p or $20/p$ when $w = 20$. Thus, when price falls to \$2, the new slope of the isoprofit curve must be $20/2 = 10$; i.e. the slope increases from 4 to 10.

Exercise 11A.12 *It appears from panel (f) of Graph 11.5 that profits are smallest (i.e. most negative) when I stop hiring at 22 labor hours per day. What can you conclude about the slope of the production frontier in panel (c) of the graph at 22 daily labor hours? Explain.*

Answer: At 22 daily labor hours, the slope of the production frontier must be equal to 20/5=4 which is equal to the slope of isoprofit curves ($w/p = 20/5$). Notice that the production frontier has two points with this slope — the profit maximizing point A where the highest possible isoprofit curve is obtained a second point on the initially non-convex portion of the production frontier. At both of these points, $w/p = MP_\ell$ because the slope of the isoprofit curve is (w/p) and the slope of the production frontier is MP_ℓ. Simply rearranging the terms in this equation, we can see that this is equivalent to saying that at both of these points $w = pMP_\ell$. In panel (f), we show that this holds at labor hours of 22 and 78 — so it must equivalently hold in panel (c) for those labor hours. Put differently, at both these labor hours there exists an isoprofit curve that is just tangent to the production frontier — but only the second of these tangencies — point A in panel (c) — is profit maximizing (because the other tangency lies on an isoprofit which is lower and thus has less profit associated with each of its production plans.)

Exercise 11A.13 *Suppose I have already signed a contract with my former student who is providing me with the factory space, machinery and raw materials for my business, and suppose that I agreed in that contract to pay my former student \$100 per month for the coming year. Is this an economic cost with respect to my decision of whether and how much to produce this year?*

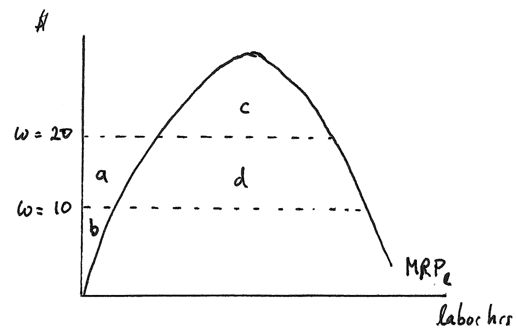
Answer: No, it is not an economic cost of producing for this year because I am legally obligated to pay it regardless of whether or how much I produce. Thus, my decision of whether or how much to produce has no impact on this “cost”.

Exercise 11A.14 *There are also production plans to the right of A where the slope of the production frontier is shallower. Why are we not considering these?*

Answer: We are not considering these because all of them would lie on isoprofit curves that fall below the magenta curve that is tangent at B — and thus all of these production plans would result in less profit than B.

Exercise 11A.15 *Which areas in the lower panel of Graph 11.6a add up to the \$200 profit I made before wages fell? Which areas add up to the \$1045 profit I make after wages fall?*

Answer: The graph is replicated below in Graph 11.3 with the critical areas labeled. The \$200 profit at the original wage of \$20 is equal to $c - a - b$. The \$1045 profit at the new wage of \$10 is equal to $c + d - b$ — which is equal to the shaded green minus the shaded magenta areas in the textbook graph.



Graph 11.3: Changing Profit as w falls

Exercise 11A.16 *Given an intercept of* -100 *of this isoprofit curve, what is the value of profit indicated by the shaded green minus the shaded magenta area in the lower panel of Graph 11.6b?*

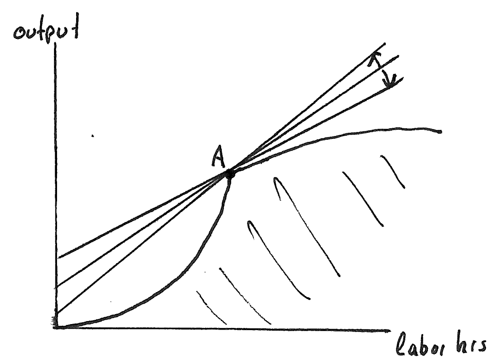
Answer: The shaded green minus the shaded magenta areas in panel (b) is equal to the profit at production plan C. The intercept of the isoprofit curve tangent at C in the top panel is -100 and is equal to Profit/p. Since the output price p is \$5, we therefore know that Profit/5=-100 — which implies that Profit=$-$\$500. Thus, the shaded green minus the shaded magenta areas in the lower part of panel (b) is $-$\$500.

Exercise 11A.17 *Had the increase in the market wage been less dramatic, would my best course of action still necessarily have been to shut down production?*

Answer: No. For smaller increases in the wage, the slope of the isoprofit curves would increase less dramatically. For sufficiently small increases in the wage, that would result in a tangency of one of the new isoprofit curves with the production frontier such that the isoprofit curve would still have positive intercept on the vertical axis — and thus the production plan at that tangency would still give positive profit. In that case, this is the optimal production plan and the firm should not shut down.

Exercise 11A.18 *What would have to be true of the production frontier in order for the original optimal production plan A to remain optimal as wages either rise somewhat or fall somewhat? (*Hint*: Consider what role kinks in the producer choice set might play.)*

Answer: In Graph 11.4, we illustrate a production frontier with a kink at A. At this kink, a number of different slopes are "tangent". Thus, beginning with wages and prices such that A is profit maximizing, we can change wages up or down within some range and still have the steeper or shallower isoprofits that result be "tangent" at A with positive intercepts on the vertical axis. In such cases, the producer would not change production plans as wages change.

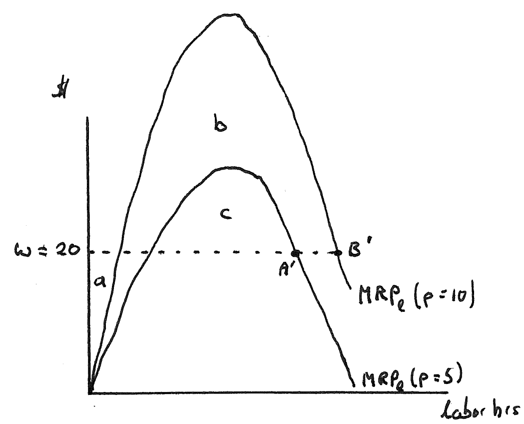


Graph 11.4: No change in profit maximizing behavior with changes in wages

Exercise 11A.19 *Why would it be economically rational for me to still stay open for business when $w = w^*$ where my profit is zero?*

Answer: When economic profit is zero, this means that the producer does as well in this activity as she could in the next best alternative activity. If she spends time on her business, this means that her labor hours are being compensated — so she is indeed "making money". But she is making just as much money as she could doing the next best thing. So, when her profit is zero, she does as well as she could anywhere else — which makes it economically rational to continue producing. (It would also in this case be economically rational to do the next best thing — which would also result in zero profit.)

Exercise 11A.20 *If I had signed a contract and agreed to make monthly payments for the next*



Graph 11.5: New Profit when p increases to \$10

year to my former students who provided me with my factory space, would w^* — the highest wage at which I will still produce — be any different?

Answer: No, w^* would be no different because I have to make the monthly payments whether I produce this year or not — which means that they are not an economic cost that is relevant for my current economic decisions.

Exercise 11A.21 *What areas in the lower panel of Graph 11.8 add up to my new profit? What is the dollar value of this new profit (which you can calculate from the intercept of the isoprofit curve in the top panel of the graph)?*

Answer: The new profit areas on the lower panel of the graph must be read off the new (magenta) MRP_ℓ curve at the unchanged wage of $w = 20$. The graph is replicated here as Graph 11.5 with some relevant areas labeled with lower case letters. The new profit is then $b + c - a$. The dollar value for this area can be calculated from the top panel of the graph in the text. The intercept of the new optimal (magenta) isoprofit (tangent at B) is Profit/p=209. Since $p = 10$, this implies Profit/10=209 or Profit=\$2,090.

Exercise 11A.22 *Can you tell from Graph 11.8 how the labor demand curve will change when p changes?*

Answer: At the lower price $p = 5$, the labor demand curve lies on the downward sloping portion of the original (blue) $MRP_\ell(p = 5)$ curve. At the new price $p = 10$, the labor demand curve lies on the downward sloping portion of the new (magenta) $MRP_\ell(p = 10)$ curve. Thus, the labor demand curve shifts out as p increases. Similarly, it would shift in when p decreases.

Exercise 11A.23 *What value would p have to take in order for isoprofits to have the same slope as when wages increased to $30 per hour (as in Graph 11.6b)? What would be my optimal course of action in that case?*

Answer: When $p = 5$ and w increases from $20 to $30, the slope of the isoprofit, which is always w/p, increases from 4 to 6. If w remained unchanged at $20, the slope w/p would change to 6 if $20/p = 6$ — i.e. if $p = 10/3 = 3.33$.

Exercise 11A.24 *In Graph 11.9 we implicitly held wage fixed. What happens to the supply curve when wage decreases?*

Answer: In panel (a), the isoprofits for any given p become shallower as w decreases — causing the tangency with the production frontier to shift up and to the right. Thus, as w decreases, output increases for any p. This then implies that the supply curve shifts to the right.

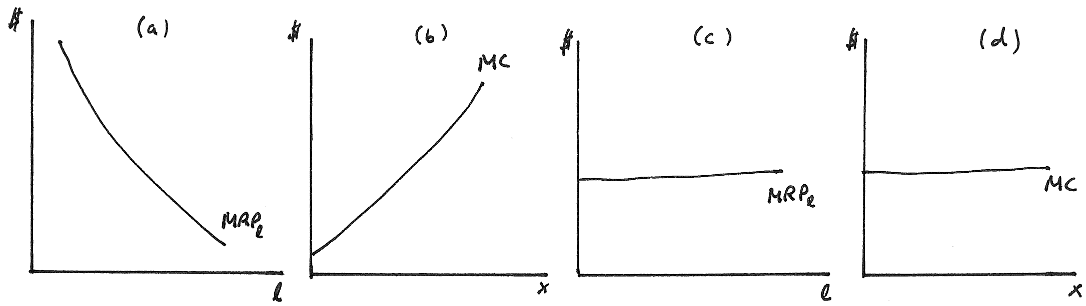
Exercise 11A.25 *If the wage rate used to construct the panels on the right of Graph 11.10 is $20, can you conclude what the slope of the production frontier in panel (a) at 10 units of output is? Can you conclude what labor input is required to produce 10 units of output, and then what the vertical values of the curves in panels (b) and (c) are for that level of labor input?*

Answer: The slope of the production frontier at 10 units of output must be 1. This is because we know from panel (e) that the marginal cost is 20 *in terms of dollars* that are used to buy the input labor — and we know labor costs $20 per unit. Thus the additional labor required to produce one more unit of x must be 1. Similarly, the total labor input required to produce 10 units of output must be 300/20=15. Since we just concluded that, when we are at output level of 10, it takes one additional unit of labor to produce 1 additional unit of output. Thus, at $\ell = 15$ (which is how much labor it takes to produce 10 units of x), the marginal product of labor must be 1 — which is then the vertical value of the MP_ℓ curve at $\ell = 15$. The vertical value for $\ell = 15$ in panel (c) is then just p.

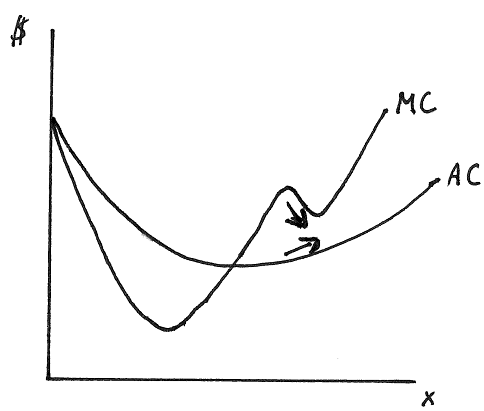
Exercise 11A.26 *What would be the shape of the MRP_ℓ and MC curves if the entire producer choice set was strictly convex? What would the shape be for the production frontier graphed in Graph 11.1(a)?*

Answer: These are illustrated in Graph 11.6 (next page) — with panels (a) and (b) illustrating the curves for strictly convex choice sets (where production becomes increasingly difficult throughout) and panels (c) and (d) illustrating the curves for the linear production constraint in Graph 11.1.

Exercise 11A.27 True or False*: On a graph with output on the horizontal and dollars on the vertical, the "marginal revenue" curve must always be a flat line so long as the producer is a price taker.*



Graph 11.6: *MRP* and *MC* Curves

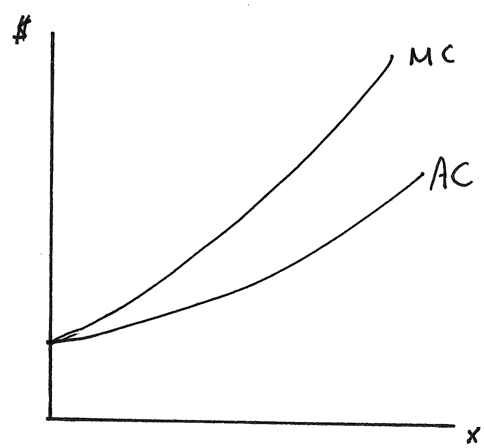


Graph 11.7: Falling *MC* with increasing *AC*

Answer: This is true. The marginal revenue is the additional revenue the producer can get from producing one more unit of output. If the producer is a price taker, she can always sell additional units of output at the market price. Thus, the marginal revenue of producing more is always equal to the market price.

Exercise 11A.28 *Can MC fall while AC rises? (*Hint*: The answer is yes.) Can you give an analogous example of marginal test grades falling while the average grade rises at the same time?*

Answer: Graph 11.7 illustrates a case where this is the case. Note that all that is required for *AC* to increase is for *MC* to *lie above AC*. It does not necessarily require *MC* to be increasing as well. So long as *MC* lies above *AC*, it drags up the *AC* even



Graph 11.8: *MC* and *AC* when the producer choice set is Convex

if, for some interval, the *MC* is actually falling. You can again think of this in terms of your grades. Suppose that, after the second exam, you have a course average of 55. Then you score a 100 on the third exam. This marginal grade of 100 raises your course average to 70. Now suppose you make a 90 on the fourth exam. Your marginal grade has fallen from 100 — but because it lies above your average (of 70) going into the exam, it will still raise your average (to 75). So, your average grade is increasing from 70 to 75 despite the fact that your marginal grade fell from 100 to 90.

Exercise 11A.29 *How do the marginal and average cost curves look if the producer choice set is convex?*

Answer: This is illustrated in Graph 11.8. The shapes should make intuitive sense — the curves must be upward sloping because a convex producer choice set implies it gets increasingly difficult to produce additional output starting from the first unit. Thus, it must mean that each additional unit is more expensive than the last. And, as always, *MC* and *AC* must start at the same point — and *MC* must lie above *AC* if *AC* is upward sloping.

11B Solutions to Within-Chapter Exercises for Part B

Exercise 11B.1 *How would this production function look differently if we did not specify that output levels off at 2α?*

Answer: The production function would have negative slope beginning at $\ell = \pi/\beta$ — and would then oscillate between positive and negative slopes as labor increases.

Exercise 11B.2 *Define the production function generating the production frontier in Graph 11.1a and define the corresponding producer choice set formally.*

Answer: The production function has constant slope and is given by $f(\ell) = 4\ell$. The producer choice set is then simply given by

$$C(f : \mathbb{R}^1_+ \to \mathbb{R}^1_+) = \left\{(x, \ell) \in \mathbb{R}^2 \mid x \leq 4\ell\right\}. \tag{11.1}$$

Exercise 11B.3 *Given that f is really defined as in equation (11.3), how should equation (11.5) be modified to accurately reflect the marginal product of labor for labor hours above 100?*

Answer: Since the marginal product of labor above 100 is zero, the full definition of the marginal product function is

$$MP_\ell = \begin{array}{ll} 6.2832 \sin(0.031416\ell) & \text{for } 0 \leq \ell \leq 100 \text{ and} \\ 0 & \text{for } \ell > 100. \end{array} \tag{11.2}$$

Exercise 11B.4 *Derive the marginal product of labor from the production function you derived in exercise 11B.2 above. Compare this to the graphical derivation in Graph 11.2a.*

Answer: Taking the derivative of the production function $f(\ell) = 4\ell$, we get $MP_\ell = 4$. This tells us that the marginal product of an additional worker is always 4 units of output — i.e. the MP curve is horizontal as derived graphically in part A of the text.

Exercise 11B.5 *Check to see that the Law of Diminishing Marginal Product (of labor) is satisfied for the production function in equation (11.3).*

Answer: The Law of Diminishing Marginal Product says that, at some point, the marginal product of labor decreases as more labor is hired — i.e. at some point, the derivative of MP_ℓ becomes negative. Recalling that the derivative of $(\sin x)$ is $(\cos x)$, we get that the derivative of MP_ℓ for this production function is

$$\frac{dMP_\ell}{dx} = 0.031416(6.2832)\cos(0.031416\ell) \approx 0.1974\cos(0.031416\ell) \qquad (11.3)$$

for $\ell < 100$ (since we defined the production function to flatten out at $\ell = 100$.) You can now check to see whether this derivative of the MP_ℓ is positive or negative for different values of ℓ by plugging in such different values between 0 and 100. You will find that the derivative is positive up to approximately $\ell = 50$ — indicating that the MP_ℓ curve initially slopes up (as in our graphs). However, as ℓ increases above 50, the derivative of MP_ℓ becomes negative — indicating that at approximately $\ell =$ 50, the MP_ℓ curve slopes down. Thus, the law of diminishing marginal product holds.

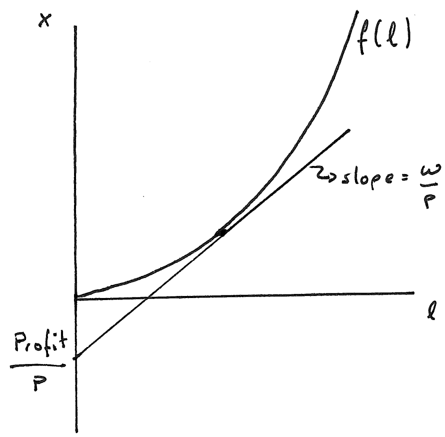
Exercise 11B.6 *Without doing the math, can you tell if the curve $\ell(p,20)$ slopes up or down? How does it relate to $\ell(p,10)$?*

Answer: We concluded in part A that, as p increases, the profit maximizing producer increases output (assuming it was optimal for her to produce to being with). With labor as the only input, this also implies that she must hire more labor. Thus, $\ell(p,20)$ is increasing in p — i.e. it slopes up. We also concluded that, as wages fall, the producer will increase output (assuming it was optimal for her to produce to begin with). Thus, when wages fall from $w = 20$ to $w = 10$, output increases and, given there is only one input available to the producer, this implies she will hire more workers. Thus, $\ell(p,20) < \ell(p,10)$ — which means that the labor demand curve shifts outward as w falls.

Exercise 11B.7 *Consider a production function that gives rise to increasing marginal product of labor throughout (beginning with the first labor hour).* True or False*: In this case, the mathematical optimization problem will unambiguously lead to a "solution" for which profit is negative.*

Answer: This is true. This is illustrated in Graph 11.9 (next page). The shape of the production function $f(\ell)$ arises from the assumption of increasing marginal product of labor throughout. This shape implies a single tangency for isoprofits with slope w/p — and this "optimal" isoprofit must then necessarily have a negative vertical intercept. Since the vertical intercept of isoprofit curves is Profit/p, and since p is always assumed to be positive, it must be that the "solution" to the optimization problem gives us a production plan that results in negative profit. The true solution is therefore not the one given by the tangency.

Exercise 11B.8 *Consider a production function that gives rise to increasing marginal product of labor throughout (beginning with the first labor hour).* True or False*: In this case, the mathematical optimization problem will give a single solution — albeit one that minimizes rather than maximizes profit.*



Graph 11.9: Mathematical "solution" when MP_ℓ is increasing throughout

Answer: It is again apparent from Graph 11.9 that, with the shape of the production function that emerges from increasing marginal product throughout, there is a single tangency for every isoprofit slope w/p. As we argued in the previous answer, this single tangency represents a production plan that gives rise to negative profit. In fact, this production plan gives the least profit (i.e. the most negative profit) of any of the possible production pans that lie on the production frontier. You can see this by simply noting that every other production plan on the frontier lies on a higher isoprofit curve — i.e. on an isoprofit parallel to the one that gives rise to the tangency but lying to the northwest. In this sense, the mathematical solution that gives the tangency production plan actually gives us the lowest possible profit conditional on producing along the production frontier (rather than inside the producer choice set.)

Exercise 11B.9 *Give an example of a producer choice set and economic conditions such that infinite production would be "optimal".*

Answer: The producer choice set and the economic conditions (i.e. wages and prices) depicted in Graph 11.9 represent such an example. Pick any point on the producer choice set and draw the isoprofit curve that goes through that point. You will be able to draw a parallel isoprofit curve that lies to the northwest (i.e. with more profit) that also intersects the production frontier — with the resulting production plan using more labor and producing more output. Since you can do this for any point on the production frontier, you can keep doing it no matter how large your production already is. Thus, you do not reach a profit maximizing production plan until you produce an infinite amount of the output.

Exercise 11B.10 *Do you think the scenario you outlined in the previous within-chapter exercise makes sense under the assumption of "price taking" behavior by producers?*

Answer: No, it does not. The price taking assumption is based on an assumption that the market contains many small producers, none of which is large enough to have any impact on price. If any producer ever became as large as the example implies, this price taking assumption no longer makes sense — the fact that the producer keeps increasing production must eventually drive up wages (as the producer is hiring more and more scarce workers) and drive down the price that the producer can charge (as consumers won't be willing to keep paying the same price no matter how much of the good is dumped on the market). The price taking assumption is therefore inconsistent with producer choice sets that give rise to increasing marginal product throughout or even increasing marginal product for large amounts of the input.

Exercise 11B.11 *What has to be true about α in order for this production function to exhibit diminishing marginal product of labor?*

Answer: The marginal product of labor is

$$MP_\ell = \frac{\partial f(\ell)}{\partial \ell} = \alpha A \ell^{\alpha-1}. \tag{11.4}$$

This is declining in labor if its derivative is less than 0; i.e. if

$$\frac{\partial MP_\ell}{\partial \ell} = (\alpha-1)(\alpha)A\ell^{\alpha-2} < 0 \tag{11.5}$$

which holds only if $\alpha < 1$.

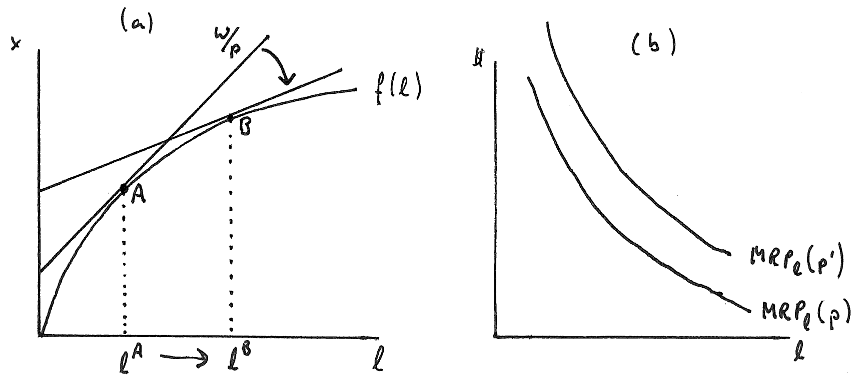
Exercise 11B.12 *Suppose $0 < \alpha < 1$ and $A > 0$. Holding price fixed, is the labor demand function downward sloping in the wage? Holding wage fixed, is it upward or downward sloping in price? Can you graphically illustrate why your answers hold?*

Answer: To determine whether the labor demand function is downward sloping in the wage, we simply need to take the derivative of the demand function with respect to wage and check whether this derivative is negative. We then get

$$\frac{\partial \ell(p,w)}{\partial w} = \left(\frac{1}{\alpha-1}\right)\left(\frac{1}{\alpha A p}\right)^{1/\alpha-1} w^{(2-\alpha)/(\alpha-1)}. \tag{11.6}$$

Given $0 < \alpha < 1$ and $A > 0$, the second and third terms in this equation are positive. The first term $(1/(\alpha-1))$, however, is negative. Thus, the derivative of the labor demand function with respect to w is negative — i.e. the labor demand function slopes down in w.

We can similarly determine which way the labor demand function slopes in p. Now we take the partial derivative of the labor demand with respect to p and check whether we can determine whether this derivative is positive or negative. The derivative is



Graph 11.10: Changes in labor demanded as w and p change

$$\frac{\partial \ell(p,w)}{\partial p} = -\left(\frac{1}{\alpha-1}\right)\left(\frac{w}{\alpha A}\right)^{1/(\alpha-1)} p^{-\alpha/(\alpha-1)}. \tag{11.7}$$

As before, the second and third terms in this equation are positive (given $0 < \alpha < 1$ and $A > 0$). The first term $1/(\alpha-1)$ is negative, but it is preceded by a negative sign which makes it positive. Thus, the partial derivative of $\ell(p,w)$ with respect to p is positive — which implies that the labor demand function is increasing (or upward sloping) in p.

Both these results make intuitive sense. The first says that an increase in the cost of inputs (i.e. labor) will cause me to hire fewer workers; while the second says that an increase in the price at which I can sell the output will cause me to hire more workers. We can illustrate this graphically as we did in part A of the text. In panel (a) of Graph 11.10 below, we illustrate the shape of the production function $f(\ell)$ which gives rise to a convex producer choice set. Suppose that w and p were initially such that the steeper isoprofit arises — giving rise to the optimal production plan A at which ℓ^A in labor hours is hired. Now suppose that w falls — causing the slope of isoprofits to fall. This results in a new profit maximizing production plan B — with ℓ^B in labor hours hired. Thus, as w falls, ℓ increases, which is the same as saying that labor demand increases as wages fall or, put differently, the labor demand function is downward sloping in w. (The labor demand function, holding p fixed, is the inverse of the labor supply curve we graph in panel (b) — but the downward slope remains as we flip the axes.) Similarly, if p increases, the isoprofit curves become shallower — again leading to an increase in labor hours hired from production plan A to B. Thus, as price increases, labor demanded goes up — i.e. the labor demand function is upward sloping in p.

The same can also be seen in the marginal revenue product graph of panel (b). Given the shape of the production function, we know that marginal product is diminishing throughout. This implies that the marginal revenue product curve is downward sloping — giving rise to the downward sloping labor demand curve and the fact that the labor demand function is downward sloping in w. At the same time, if p increases (from p to p'), then the marginal revenue product curve shifts up. Thus, for any given wage, more labor is hired as p increases — i.e. labor demand is upward sloping in p.

Exercise 11B.13 *Suppose* $0 < \alpha < 1$ *and* $A > 0$. *Holding wage fixed, is the supply function upward sloping in price? Holding price fixed, is the supply function upward sloping in wage? Can you graphically illustrate why your answers hold?*

Answer: To see whether the supply function is upward or downward sloping in price, we have to take the derivative of $x(p,w)$ with respect to p and check whether this derivative is positive or negative. This derivative is

$$\frac{\partial x(p,w)}{\partial p} = -\left(\frac{\alpha}{\alpha-1}\right)\left[A\left(\frac{w}{\alpha A}\right)^{\alpha/(\alpha-1)}\right]p^{(1-2\alpha)/(\alpha-1)}. \tag{11.8}$$

Since $0 < \alpha < 1$ and $A > 0$, the bracketed term is positive, as is the last term in the equation. This leaves the first term $\alpha/(\alpha-1)$ which is negative. However, it is preceded by a negative sign — so the whole term is therefore positive. Thus, $x(p,w)$ is increasing in p — which is to say that the supply function is upward sloping in p.

We can similarly check whether the supply function is upward or downward sloping in w by considering the derivative

$$\frac{\partial x(p,w)}{\partial w} = \frac{\alpha}{\alpha-1}\left[A\left(\frac{1}{\alpha Ap}\right)^{\alpha/(\alpha-1)}\right]w^{1/(\alpha-1)}. \tag{11.9}$$

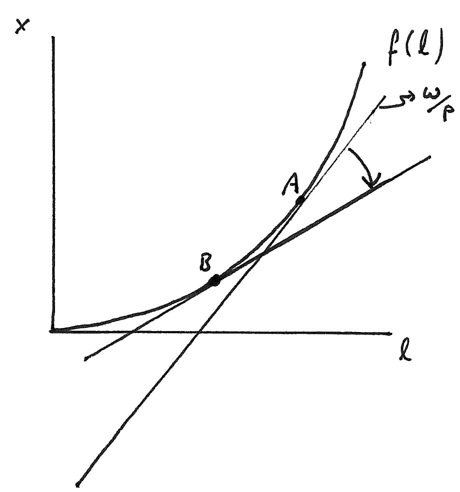
Again, the bracketed and the last term are positive, but the first term $\alpha/(\alpha-1)$ is negative since $0 < \alpha < 1$. As a result, the whole derivative is negative, which implies that the output supply function is decreasing, or downward sloping, in w.

Both these results again make intuitive sense. The first says that, as output price increases, the profit maximizing supply quantity increases as well. The second says that, as input price (wage) decreases, the profit maximizing output quantity increases. This can also be seen in panel (a) of Graph 11.10. Suppose we start at prices and wages such that A is the profit maximizing production plan. An increase in p and a decrease in w both have the effect of decreasing w/p — the slope of the isoprofit curves. Thus, both result in increased output to a new profit maximizing production plan such as B.

Exercise 11B.14 *How do your answers to the previous two exercises change when* $\alpha > 1$*? Can you make sense of what is going on? (*Hint*: Graph the production function and illustrate the tangencies of isoprofits for different wages and prices.)*

Answer: The mathematical answers would reverse entirely — with labor demand now increasing in w and decreasing in p, and with output supply increasing in w and decreasing in p. This does not make intuitive sense. An increase in output price should cause producers to increase production and hire more labor, not the other way around; and an increase in input price (wage) should cause a decrease in production and less labor demanded. The reason that the math is now giving the wrong answer is because the "right" answer would be a corner solution. This is because, when $\alpha > 1$, the production function has increasing slope — implying increasing (rather than diminishing) marginal product of labor.

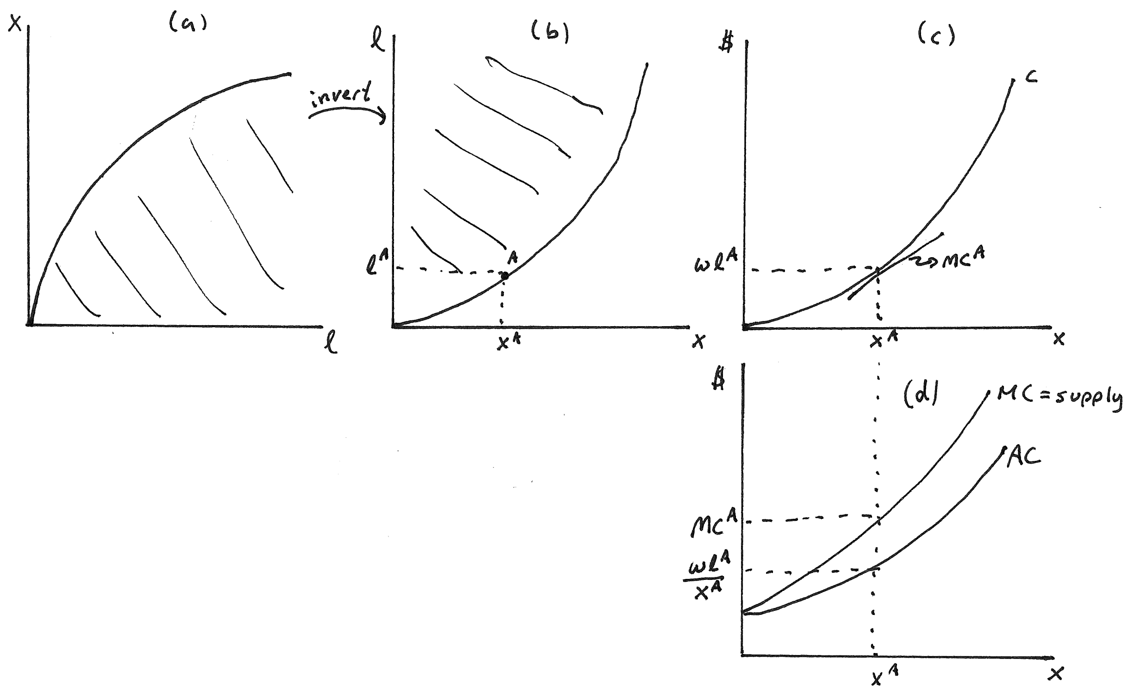
To see what is going on, consider Graph 11.11. Here we graph a production function with the slope that $\alpha > 1$ implies. Suppose we initially face wage w and price p, giving rise to the isoprofit that is tangent at A. This is the production plan that the math gives us as the "solution". The problem, of course, is that profit is negative under this plan (as indicated by the negative vertical intercept of the isoprofit curve). Now suppose that w falls — which causes isoprofits to become shallower (given that their slope is w/p). The new production plan found by the mathematical optimization method we used is now B — indicating less labor input and less output supply. Thus, the math suggests that a drop in the input price (wage) results in less labor demanded and less output supplied, exactly counter to what should be true. Again, of course, profit at B is negative — so neither A nor B are truly profit maximizing production plans, which leads to the nonsensical result. Similarly, suppose p increases from the original. This similarly causes w/p to fall — making isoprofits shallower. Again we move from the "solution" A to the "solution" B — suggesting that an increase in output price causes the producer to produce less and hire fewer workers. And again the result is nonsensical because neither the original plan A nor the new plan B are actually profit maximizing. In fact, in both cases the producer should produce an infinitely large quantity.



Graph 11.11: Nonsensical "solutions" when $\alpha > 1$

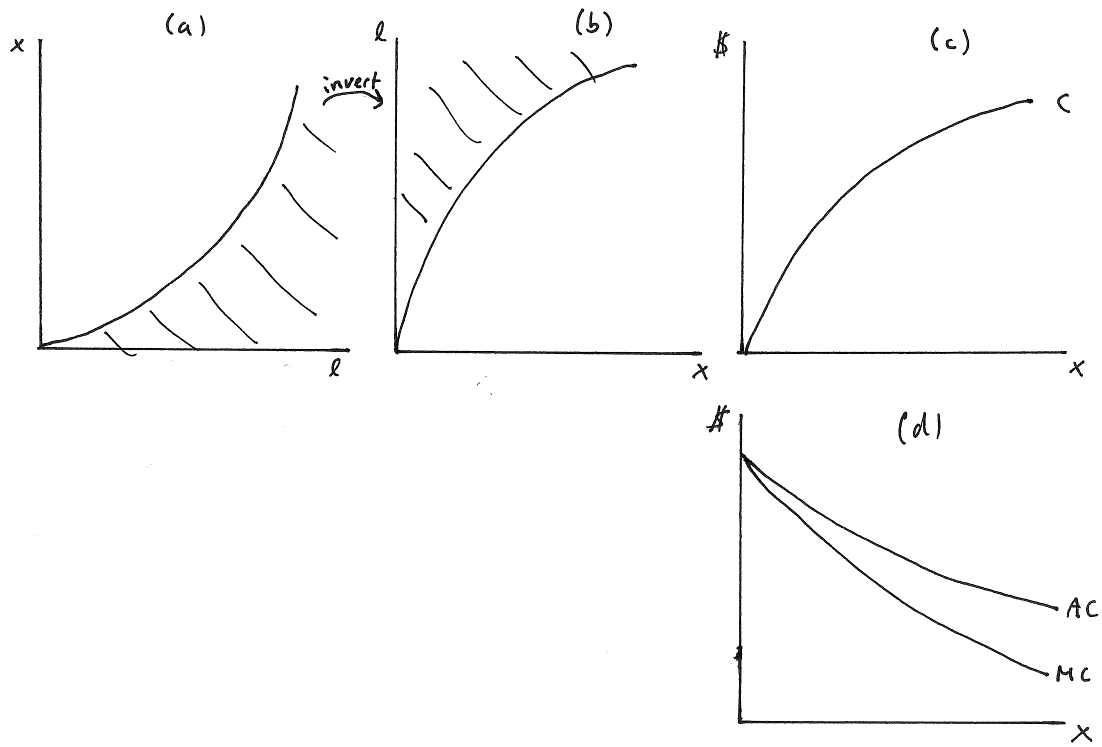
Exercise 11B.15 *Graphically illustrate the way we have just derived the output supply function assuming α lies between 0 and 1. What changes when $\alpha > 1$?*

Answer: In Graph 11.12, we start with a convex production set in panel (a) (which emerges when $\alpha < 1$). This is inverted in panel (b) — which corresponds to the step where we invert the production function and write it in terms of $\ell(x)$. We then multiply this by w to get the cost function in panel (c) (graphed with w held fixed) — then take the derivative to get MC in (d) and we divide the cost function by x to get the AC in panel (d).



Graph 11.12: Solution when $\alpha > 1$

When $\alpha > 1$, the producer choice set is non-convex as illustrated in panel (a) of Graph 11.13 (next page). We can again invert the production to get it into the form $\ell(x)$ that tells us how much labor is required for each level of output — then multiply by w to get the cost function that is illustrated in panel (c) with wage held fixed. Then we get the MC and AC the same way as before in panel (d). The difference is that we cannot now claim that the MC function is the supply curve — because the supply curve has to lie above AC.



Graph 11.13: Nonsensical "solutions" when $\alpha > 1$

11C Solutions to End-of-Chapter Exercises

Exercise 11.3

Consider a profit maximizing firm.

A: *Explain whether the following statements are true or false:*

(a) *For price-taking, profit maximizing producers, the "constraint" is determined by the technological environment in which the producer finds herself while the "tastes" are formed by the economic environment in which the producer operates.*

Answer: The physical constraint that a producer cannot get around arises from the simple laws of physics — you can only get so much x out of the inputs you use. The more sophisticated the technology available to the producer, the more x she can squeeze out — and thus the technology creates the production constraint the tells the producer which production plans are feasible and which are not. In terms of tastes, we typically assume that producers simply care about profit — and that more profit is better than less. To form indifference curves for producers who simply care about profit, we have to find production plans that all result in the same amount of profit. And how easy it is to make profit depends on how high the output price is and how low the input prices are relative to the output price. Thus, the production plans that produce the same level of profit will differ depending on the economic environment — depending on how much the producer can get for her output and how much she has to pay for the inputs in her production plans.

(b) *Every profit maximizing producer is automatically cost-minimizing.*

Answer: This is true. If you are maximizing profit, you must be producing whatever you are producing without wasting any inputs — i.e. you must be producing the output at the minimum cost possible.

(c) *Every cost-minimizing producer is automatically profit maximizing.*

Answer: This is not true. You could be producing some arbitrary quantity without wasting any inputs — i.e. in the least cost way. But that does not mean you are producing the "right" quantity. Cost minimization is only part of profit maximization — it only takes into account input prices as they relate to the cost of production. Only when you profit maximize do you also take into account the output price and what that implies for how much you should produce in a cost minimizing way.

(d) *Price taking behavior makes sense only when marginal product diminishes at least at some point.*

Answer: This is true. If marginal product always increases, then it is getting cheaper and cheaper to produce additional units of output. And if I can sell all my output at the same price (i.e. if I am a price taker), then I should keep producing and drive down my average cost.

B: *Consider the production function $x = f(\ell) = \alpha\ln(\ell+1)$.*

(a) *Does this production function have increasing or decreasing marginal product of labor.*

Answer: Marginal product for this production function is given by

$$MP_\ell = \frac{\alpha}{\ell-1}. \tag{11.10}$$

Thus, as ℓ increases, MP_ℓ clearly decreases. You can also see this by simply taking the derivative of marginal product of labor with respect to ℓ —

$$\frac{\partial MP_\ell}{\partial \ell} = \frac{-\alpha}{(\ell-1)^2}. \tag{11.11}$$

(b) *Set up the profit maximization problem and solve for the labor input demand and output supply functions.*

Answer: The profit maximization problem is

$$\max_{\ell,x}\ px - w\ell \ \text{ subject to } \ x = \alpha\ln(\ell+1) \tag{11.12}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell}\ p\alpha\ln(\ell+1) - w\ell. \tag{11.13}$$

The first order condition for this problem is

$$\frac{p\alpha}{\ell+1} - w = 0 \tag{11.14}$$

which can be solved to get the labor demand function

$$\ell(w,p) = \frac{\alpha p - w}{w}. \tag{11.15}$$

Substituting this into the production function, we then get the output supply function

$$x(w,p) = \alpha\ln\left(\frac{p\alpha - w}{w} + 1\right) = \alpha\ln\left(\frac{\alpha p}{w}\right). \tag{11.16}$$

(c) *Recalling that* $\ln x = y$ *implies* $e^y = x$ *(where* $e \approx 2.7183$ *is the base of the natural log), invert the production function and derive from this the cost function* $c(w, x)$.

Answer: To invert the production function $x = \alpha \ln(\ell + 1)$, we first note that this implies

$$e^{x/\alpha} = \ell + 1 \tag{11.17}$$

which can then be solved in terms of ℓ to give us the amount of labor required for each level of output; i.e.

$$\ell = e^{x/\alpha} - 1. \tag{11.18}$$

When multiplied by w, we then get the cost function

$$c(w, x) = w\left(e^{x/\alpha} - 1\right). \tag{11.19}$$

(d) *Determine the marginal and average cost functions.*

Answer: The marginal cost is

$$MC(w, x) = \frac{\partial c(w, x)}{\partial x} = \frac{w}{\alpha} e^{x/\alpha}, \tag{11.20}$$

and the average cost is

$$AC(w, x) = \frac{we^{x/\alpha}}{x} - \frac{w}{x}. \tag{11.21}$$

(e) *Derive from this the output supply and labor demand functions. Compare them to what you derived directly from the profit maximization problem in part (b).*

Answer: To derive the output supply function, we begin by setting p equal to MC; i.e.

$$p = \frac{w}{\alpha} e^{x/\alpha}. \tag{11.22}$$

Multiplying both sides by α/w, taking natural logs and then multiplying both sides by α, we then get

$$x(p, w) = \alpha \ln \frac{\alpha p}{w}. \tag{11.23}$$

Plugging this back into equation (11.18), we get the labor input demand

$$\ell(w, p) = e^{\ln(\alpha p/w)} - 1 = \frac{\alpha p}{w} - 1 = \frac{\alpha p - w}{w}. \tag{11.24}$$

Note that the output supply and labor demand equations are identical to those derived directly through profit maximization earlier in the problem. The equations are correct, however, only for prices above the lowest point of AC.

(f) *In your mathematical derivations, what is required for a producer to be cost minimizing? What, in addition, is required for her to be profit maximizing?*

Answer: The only requirement for the producer to be cost minimizing is that she not waste any input — i.e. that she hire labor in accordance with equation (11.18). There is no requirement imposed by cost minimization on *how much* to produce. The additional requirement imposed by profit maximization is that output quantity be chosen such that $p = MC$. Alternatively, we could phrase this same additional requirement as the requirement that $pMP_\ell = w$.

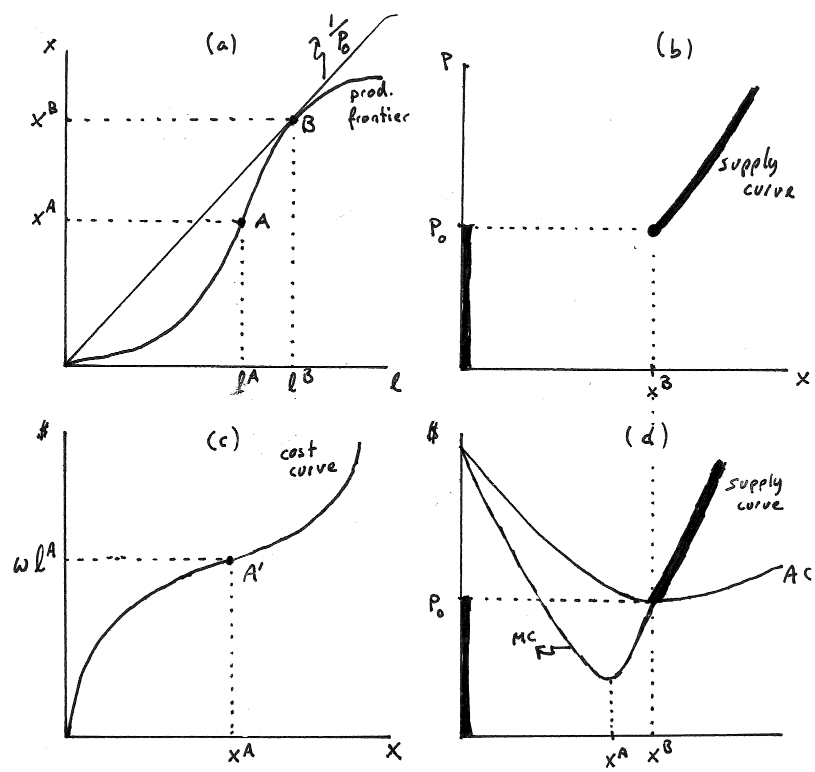
Exercise 11.7

We have shown that there are two ways in which we can think of the producer as maximizing profits: Either directly, or in a two-step process that begins with cost minimization.

A: *This exercise reviews this equivalence for the case where the production process initially has increasing marginal product of labor but eventually reaches decreasing marginal product. Assume such a production process throughout.*

(a) *Begin by plotting the production frontier with labor on the horizontal and output on the vertical axis. Identify in your graph the production plan $A = (\ell^A, x^A)$ at which increasing returns turns to decreasing returns.*

<u>Answer</u>: This is illustrated in panel (a) of Graph 11.14 (next page) where A lies at the point at which the production frontier stops increasing at an increasing rate and starts increasing at a decreasing rate. Put differently, at A the slope stops increasing and starts decreasing.



Graph 11.14: 2 Ways to Maximize Profit

(b) *Suppose wage is $w = 1$. Illustrate in your graph the price p_0 at which the firm obtains zero profit by using a production plan B. Does this necessarily lie above or below A on the production frontier?*

<u>Answer</u>: This is also illustrated in panel (a) where B lies on an isoprofit that is tangent to the production frontier at B *and* intersects the origin (which implies zero profit). The slope of the isoprofit is $1/p_0$ since $w = 1$. It is apparent from the graph that B must lie above A on the production frontier — i.e. it must be the case that the zero-profit price results in production on the decreasing marginal product of labor portion of the production frontier.

(c) *Draw a second graph next to the one you have just drawn. With price on the vertical axis and output on the horizontal, illustrate the amount the firm produces at p_0.*

Answer: This is illustrated in panel (b) of Graph 11.14 where the point (x^B, p_0) illustrates the lowest price at which the firm produces positive output.

(d) *Suppose price rises above p_0. What changes on your graph with the production frontier — and how does that translate to points on the supply curve in your second graph?*

Answer: If price rises above p_0, the isoprofit lines become shallower — which implies the new optimal quantity lies at a tangency higher on production frontier than B. The isoprofit that is tangent at the new profit maximizing production plan also has positive intercept on the vertical axis — implying profit will be positive. Thus, output increases as p rises above p_0 — leading to an upward sloping supply curve (as illustrated in panel (b).)

(e) *What if price falls below p_0?*

Answer: If price falls below p_0, the isoprofit curves become steeper — implying tangencies to the left of B. At those tangencies, however, the intercept on the vertical axis will be negative — implying negative profit. Thus, the firm is better off not producing at all — which is why the supply curve in panel (b) is vertical at zero output level up to the price p_0.

(f) *Illustrate the cost curve on a graph below your production frontier graph. What is similar about the two graphs — and what is different — around the point that corresponds to production plan A.*

Answer: The cost curve, as illustrated in panel (c) of Graph 11.14, has the inverse shape from the production frontier — because when each additional labor unit increases production more than the last (on the increasing marginal product part of the production frontier), the cost of increasing output rises at a decreasing rate (and vice versa). Around A' — the point corresponding to A in panel (a), the cost curve switches from increasing at a decreasing rate to increasing at an increasing rate (just as the switch from increasing at an increasing rate to increasing at a decreasing rate happens on the production frontier.)

(g) *Next to your cost curve graph, illustrate the marginal and average cost curves. Which of these reaches its lowest point at the output quantity x^A? Which reaches its lowest point at x^B?*

Answer: This is illustrated in panel (d) of Graph 11.14. The marginal cost (MC) curve is the slope of the cost curve — so it reaches its lowest point at the output level x^A where the slope of the cost curve begins to get steeper. The average cost curve reaches its lowest point where the MC curve crosses it — which is also where the supply curve begins. This occurs at x^B.

(h) *Illustrate the supply curve on your graph and compare it to the one you derived in parts (c) and (d).*

Answer: The supply curve is the part of the MC curve that lies above the AC curve — with output of zero below that. This is highlighted in panel (d) of Graph 11.14 — with the resulting supply curve being identical to what we derived before in panel (b).

B: *Suppose that you face a production technology characterized by the function $x = f(\ell) = \alpha/(1 + e^{-(\ell-\beta)})$.*

(a) *Assuming labor ℓ costs w and the output x can be sold at p, set up the profit maximization problem.*

Answer: The profit maximization problem is

$$\max_{\ell,x}\ px - w\ell \ \text{ subject to } \ x = \frac{\alpha}{1+e^{-(\ell-\beta)}} \tag{11.25}$$

which can be written as the unconstrained problem

$$\max_{\ell}\ \frac{p\alpha}{1+e^{-(\ell-\beta)}} - w\ell. \tag{11.26}$$

(b) *Derive the first order condition for this problem.*

Answer: The first order condition is

$$\frac{\alpha p e^{-(\ell-\beta)}}{\left[1+e^{-(\ell-\beta)}\right]^2} = w. \tag{11.27}$$

(c) *Substitute $y = e^{-(\ell-\beta)}$ into your first order condition and, using the quadratic formula, solve for y. Then, recognizing that $y = e^{-(\ell-\beta)}$ implies $\ln y = -(\ell-\beta)$, solve for the two implied labor inputs and identify which one is profit maximizing (assuming that an interior production plan is optimal).*

Answer: Substituting $y = e^{-(\ell-\beta)}$, the first order condition reduces to $\alpha p y/(1+y)^2 = w$ which can be written in the form

$$wy^2 + (2w - \alpha p)y + w = 0. \tag{11.28}$$

The quadratic formula then gives two solutions for y:

$$y_1 = \frac{-(2w-\alpha p) + \sqrt{(2w-\alpha p)^2 - 4w^2}}{2w} = \frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w} \tag{11.29}$$

and

$$y_2 = \frac{-(2w-\alpha p) - \sqrt{(2w-\alpha p)^2 - 4w^2}}{2w} = \frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}. \tag{11.30}$$

Given that $y = e^{-(\ell-\beta)}$, we can take natural logs of both sides to get $-(\ell-\beta) = \ln y$ or $\ell = \beta - \ln y$. Using the two solutions for y, we therefore get

$$\begin{aligned} \ell_1 &= \beta - \ln\left(\frac{\alpha p - 2w + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right) \quad \text{and} \\ \ell_2 &= \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right). \end{aligned} \tag{11.31}$$

Since $y_1 > y_2$, we know that $\ell_1 < \ell_2$ — and thus the true profit maximizing labor input (assuming an interior production plan is profit maximizing) is given by

$$\ell(w,r,p) = \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right). \tag{11.32}$$

(d) *Use your answer to solve for the supply function (assuming an interior solution is optimal).*

Answer: Plugging $\ell = \beta - \ln y_2$ into the production function, we then get

$$x = \frac{\alpha}{1 + e^{-(\beta - \ln y_2 - \beta)}} = \frac{\alpha}{1 + e^{\ln y_2}} = \frac{\alpha}{1 + y_2}. \tag{11.33}$$

Substituting in y_2 from equation (11.30), this then simplifies to the supply function (assuming an interior optimum)

$$x(w,r,p) = \frac{2\alpha w}{\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p}}. \tag{11.34}$$

We can also re-write this by multiplying both numerator and denominator by the term $(\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p})$ to get

$$x(w,r,p) = \frac{2\alpha w\left(\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}\right)}{\alpha^2 p^2 - (\alpha^2 p^2 - 4\alpha w p)} = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}. \tag{11.35}$$

(e) *Now use the two-step method to verify your answer. Begin by solving the production function for ℓ to determine how much labor is required for each output level assuming none is wasted.*

Answer: To do the 2-step optimization, we begin by solving the production function $x = \alpha/(1+e^{-(\ell-\beta)})$ for ℓ. We can do this by first multiplying through by $(1+e^{-(\ell-\beta)})$, dividing by x and subtracting 1 from both sides to get

$$e^{-(\ell-\beta)} = \frac{\alpha}{x} - 1 = \frac{\alpha - x}{x} \tag{11.36}$$

which allows us to write

$$-(\ell - \beta) = \ln\left(\frac{\alpha - x}{x}\right) \tag{11.37}$$

which then solves to

$$\ell = \beta - \ln\left(\frac{\alpha - x}{x}\right). \tag{11.38}$$

(f) *Use your answer to derive the cost function and the marginal cost function.*

Answer: The minimum cost at which the firm can produce any level of output x is then simply this inverted production function times the wage; i.e. the cost function is

$$C(w,x) = w\beta - w\ln\left(\frac{\alpha - x}{x}\right). \tag{11.39}$$

From this, we can get the marginal cost function

$$MC(w,x) = \frac{\partial C}{\partial x} = \frac{\alpha w}{(\alpha - x)x}. \tag{11.40}$$

(g) *Set price equal to marginal cost and solve for the output supply function (assuming an interior solution is optimal). Can you get your answer into the same form as the supply function from your direct profit maximization problem?*

Answer: Setting $MC = p$, we can then solve for the supply function; i.e. we set $p = \alpha w/((\alpha - x)x)$, multiply through and write it in the form that allows us to once again apply the quadratic formula:

$$x^2 - \alpha x + \frac{\alpha w}{p} = 0 \quad \text{or equivalently} \quad px^2 - \alpha p x + \alpha w = 0. \tag{11.41}$$

Applying the quadratic formula, we get two "solutions":

$$x_1 = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \quad \text{and} \quad x_2 = \frac{\alpha p - \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \tag{11.42}$$

of which the first one is the true solution (since it is the larger of the two). The supply function (assuming an interior solution) is therefore

$$x(w,r,p) = \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4p\alpha w}}{2p} \tag{11.43}$$

which is equivalent to the previous solution we got in equation (11.35) by solving the profit maximization problem directly.

(h) *Use the supply function and your answer from part (e) to derive the labor input demand function (assuming an interior solution is optimal). Is it the same as what you derived through direct profit maximization in part (c)?*

Answer: Plugging the supply function into equation (11.38), we get

$$\ell = \beta - \ln\left[\frac{\alpha - \frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}}{\frac{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2p}}\right] = \beta - \ln\left[\frac{\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{\alpha p + \sqrt{\alpha^2 p^2 - 4\alpha w p}}\right]. \tag{11.44}$$

Multiplying both the denominator and numerator within the brackets by the numerator $(\alpha p - \sqrt{\alpha^2 p^2 - 4\alpha w p})$, we can then write this as

$$\begin{aligned}\ell &= \beta - \ln\left[\frac{\alpha^2 p^2 - 2\alpha p\sqrt{\alpha^2 p^2 - 4\alpha w p} + (\alpha^2 p^2 - 4\alpha w p)}{\alpha^2 p^2 - (\alpha^2 p^2 - 4\alpha w p)}\right] \\ &= \beta - \ln\left(\frac{\alpha p - 2w - \sqrt{\alpha^2 p^2 - 4\alpha w p}}{2w}\right)\end{aligned} \qquad (11.45)$$

which is exactly equal to the labor demand function we derived in equation (11.32) through direct profit maximization.

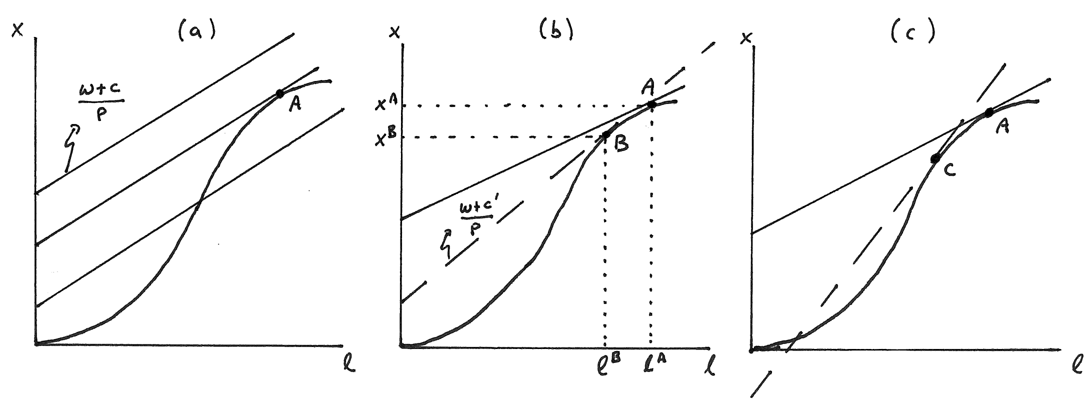
Exercise 11.10: *Optimal Response to Labor Regulations*

Business Application: *Optimal Response to Labor Regulations*: *Governments often impose costs on businesses in direct relation to how much labor they hire. They may, for instance, require that businesses provide certain benefits like health insurance or retirement plans.*

A: *Suppose we model such government regulations as a cost c per worker in addition to the wage w that is paid directly to the worker. Assume that you face a production technology that has the typical property of initially increasing marginal product of labor that eventually diminishes.*

(a) *Illustrate the isoprofits for this firm and include both the explicit labor cost w as well as the implicit cost c of the regulation.*

Answer: Three isoprofits are illustrated in panel (a) of Graph 11.15. The only difference from the usual case is that we must include c as part of the labor cost to the firm — thus causing the slope of the isoprofit curves to be $(w+c)/p$.



Graph 11.15: Increasing Regulatory Labor Costs

(b) *Illustrate the profit maximizing production plan.*

Answer: This is illustrated as A in panel (a) of the graph.

(c) *Assuming that it continues to be optimal for your firm to produce, how does your optimal production plan change as c increases?*

Answer: When c increases to c', the slope of the isoprofits get steeper. Thus, the optimal production plan in panel (b) of Graph 11.15 changes from A to B — less labor input and lower output.

(d) *Illustrate a case where an increase in c is sufficiently large to cause your firm to stop producing.*

Answer: This is illustrated in panel (c) where the new (dashed) isoprofit has become sufficiently steep as a result of an increase in c such that the "optimal" production plan C lies on an isoprofit with negative vertical intercept — and thus negative profit. As a result, this firm would not produce at C but would simply choose to hire no labor and produce no output.

(e) True or False*: For firms that make close to zero profit, additional labor regulations might cause large changes in behavior.*

Answer: This is true. When profit is high, the regulatory cost associated with labor can be large without causing profit to fall to zero. While this would still cause a change in firm behavior (as illustrated in panel (b)), it would be a marginal change — somewhat less labor and somewhat less output. But if profit initially is close to zero, then even a small increase in regulatory labor costs can cause the firm to shut down completely — and thus cause a dramatic change in behavior.

B: *Suppose that your production technology can be represented by the production function $x = 100/(1+e^{-(\ell-5)})$ where $e \approx 2.7183$ is the base of the natural logarithm.*

(a) *Suppose $w = 10$ and $p = 1$. Set up your profit maximization problem and explicitly include the cost of regulation.*

Answer: The profit maximization problem is

$$\max_{x,\ell}\; x-(10+c)\ell \;\text{ subject to }\; x=\frac{100}{1+e^{-(\ell-5)}} \tag{11.46}$$

which can also be written as the unconstrained optimization problem

$$\max_{\ell}\; \frac{100}{1+e^{-(\ell-5)}}-(10+c)\ell. \tag{11.47}$$

(b) *Calculate the optimal labor demand and output supply as a function of c. (*Hint*: Solving the first order condition becomes considerably easier if you substitute $y = e^{-(\ell-5)}$ and solve for y using the quadratic formula. Once you have a solution for y, you know this is equal to $e^{-(\ell-5)}$. You can then take natural logs of both sides, recalling that $\ln e^{-(\ell-5)} = -(\ell-5)$. This follows the steps in exercise **??** where we used an almost identical production function.)*

Answer: The first order condition for the unconstrained problem above is

$$\frac{100e^{-(\ell-5)}}{\left(1+e^{-(\ell-5)}\right)^2}-(10+c)=0. \tag{11.48}$$

Define $y=e^{-(\ell-5)}$. Substituting this into the above, we get

$$\frac{100y}{(1+y)^2}-(10+c)=0. \tag{11.49}$$

Multiplying out the denominator and re-arranging terms, we then get

$$(10+c)y^2+(2c-80)y+(10+c)=0. \tag{11.50}$$

For an equation of the form $ax^2+bx+c=0$, the quadratic formula gives the solutions

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \;\text{ and }\; x=\frac{-b-\sqrt{b^2-4ac}}{2a}. \tag{11.51}$$

We can thus solve for y by applying this formula (letting $a=(10+c)$, $b=(2c-80)$ and $c=(10+c)$) to get two candidate solutions:

$$y = \frac{80-2c+\sqrt{(2c-80)^2-4(10+c)^2}}{2(10+c)} \text{ and } y = \frac{80-2c-\sqrt{(2c-80)^2-4(10+c)^2}}{2(10+c)}. \quad (11.52)$$

Taking natural logs of $y = e^{-(\ell-5)}$, we can write $-(\ell-5) = \ln y$ where y can take one of the two values we solved for above. Solving for ℓ, we get $\ell = 5 - \ln y$. The larger of the two solutions that this provides is the profit maximizing quantity (assuming it yields positive profit).

(c) *What is the profit maximizing production plan when $c = 0$?*

Answer: When $c = 0$, the two solutions for y reduce to $y \approx 7.873$ and $y \approx 0.127$. The corresponding solutions for ℓ are then

$$\ell = 5 - \ln(7.873) = 2.937 \text{ and } \ell = 5 - \ln(0.127) = 7.063. \quad (11.53)$$

Of these, the larger is the true profit maximum — and plugging this into the production function, it gives output of $x = 88.73$.

(d) *How does your answer change when $c = 2$?*

Answer: The two solutions for y are now 6.171 and 0.162 — with the latter resulting in the higher labor input of $\ell = 6.82$ and output of $x = 86.056$.

(e) *What if $c = 3$? (*Hint*: Check to see what happens to profit.)*

Answer: The two solutions for y are now 5.511 and 0.181 — with the latter resulting in the higher labor input of $\ell = 6.707$ and output of $x = 84.641$. However, when we evaluate profit at this production plan, we get

$$\text{Profit} = 84.641 - (10+3)6.707 = -2.55. \quad (11.54)$$

Since profit at the "optimal" production plan is negative, the true optimal production plan is hiring no labor and producing no output (which results in zero profit). (You can check that for $c = 0$ and $c = 2$ in the previous parts, profit for producing the production plans we identified is in fact positive.)

Exercise 11.13: *Determining Optimal Class Size*

Policy Application: *Determining Optimal Class Size*: *Public policy makers are often pressured to reduce class size in public schools in order to raise student achievement.*

A: *One way to model the production process for student achievement is to view the "teacher/student" ratio as the input. For purposes of this problem, let t be defined as the number of teachers per 1000 students; i.e. $t = 20$ means there are 20 teachers per 1,000 students. Class size in a school of 1000 students is then equal to $1000/t$.*

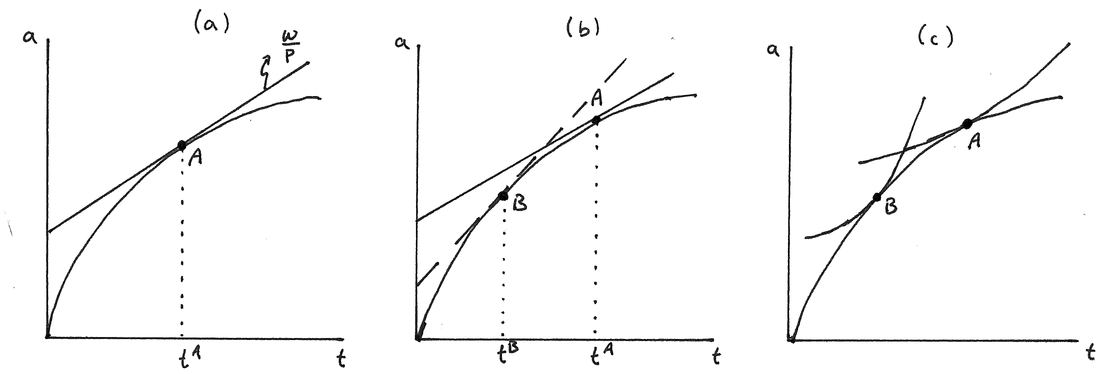
(a) *Most education scholars believe that the increase in student achievement from reducing class size is high when class size is high but diminishes as class size falls. Illustrate how this translates into a production frontier with t on the horizontal axis and average student achievement a on the vertical.*

Answer: This production frontier is pictured in panel (a) of Graph 11.16 (next page) — with large slope (or marginal product) initially that falls as t increases. Note that t is not class size — when t is small, there are few teachers per 1000 students, which implies a large class size. In our graph, class size therefore falls as t increases.

(b) *Consider a school with 1,000 students. If the annual salary of a teacher is given by w, what is the cost of raising the input t by 1 — i.e. what is the cost per unit of the input t?*

Answer: Since t is the number of teachers per 1000 students, we need to hire one more teacher to increase t by 1. Thus, the per unit cost of t is w.

(c) *Suppose a is the average score on a standardized test by students in the school, and suppose that the voting public is willing to pay p for each unit increase in a. Illustrate the "production*



Graph 11.16: Setting Class Size

plan" that the local school board will choose if it behaves analogously to a profit maximizing firm.

Answer: This is also illustrated in panel (a) of Graph 11.16 where A is the optimal "production plan" given the willingness p of voters to get a one unit increase in achievement. The input t^A then results in class size of $1000/t^A$.

(d) *What happens to class size if teacher salaries increase?*

Answer: If teacher salaries increase, w goes up — which implies w/p increases and the isoprofits become steeper. As a result, the optimal t falls — which implies class size increases. This is illustrated in panel (b) of Graph 11.16.

(e) *How would your graph change if the voting public's willingness to pay per unit of a decreases as a increases?*

Answer: This would in effect imply that p is high when a is low but decreases as a increases. A high p implies a relatively flat slope — thus, we would get isoprofit curves that, rather than being straight lines as in panels (a) and (b) of the graph, would be shaped as depicted in panel (c). (Note: The isoprofit tangent at B represents the curve when teacher salaries are higher.)

(f) *Now suppose that you are analyzing two separate communities that fund their equally sized schools from tax contributions by voters in each school district. They face the same production technology, but the willingness to pay for marginal improvements in a is lower in community 1 than in community 2 at every production plan. How do the isoprofit maps differ for the two communities?*

Answer: At every "production plan" the isoprofits of the community with lower willingness to pay would be steeper — indicating that a greater increase in achievement would be required for every increase in t in order for the community to remain equally well off.

(g) *Illustrate how this will result in different choices of class size in the two communities.*

Answer: This can be illustrated in a graph identical to the one pictured in panel (c) of Graph 11.16 where the community with the lower willingness to pay optimizes at B while the other community optimizes at A.

(h) *Suppose that the citizens in each of the two communities described above were identical in every way except that those in community 1 have a different average income level than those in community 2. Can you hypothesize which of the two communities has greater average income?*

Answer: Community 1 is poorer. Even though they care as much about education, their willingness to pay is lower simply because they have fewer resources. (This should be familiar from consumer theory — people with identical tastes can have very different demands for goods if they have different incomes.)

(i) *Higher level governments often subsidize local government contributions to public education, particularly for poorer communities. What changes in your picture of a community's optimal class size setting when such subsidies are introduced?*

Answer: You can think of these subsidies as either reducing the teacher salaries w (since the higher level government now shares in the expense) or as raising the willingness to pay p since the voters know that they get more resources if they spend more on education. Whether viewed as a decrease in w or and increase in p, the impact on w/p is the same: w/p falls — which implies that isoprofit curves become shallower. This in turn has the impact of increasing the optimal choice of t (as illustrated in panels (b) and (c) of Graph 11.16) — which is the same as saying that class size will decrease.

B: *Suppose the production technology for average student achievement is given by $a = 100t^{0.75}$, and suppose again that we are dealing with a school that has 1000 students.*

(a) *Let w denote the annual teacher salary in thousands of dollars and let p denote the community's marginal willingness to pay for an increase in student achievement. Calculate the "profit maximizing" class size.*

Answer: We need to solve the maximization problem

$$\max_{a,t} \; pa - wt \;\; \text{subject to} \;\; a = 100t^{0.75} \tag{11.55}$$

which can be written as the unconstrained optimization problem

$$\max_{t} \; a = 100t^{0.75}p - wt. \tag{11.56}$$

The first order condition for this problem is $75t^{-0.25}p - w = 0$ and can be solved to get

$$t = \left(\frac{75p}{w}\right)^4. \tag{11.57}$$

Since class size is $1000/t$, the optimal class size is

$$\text{Class size} = \frac{1000w^4}{(75p)^4}. \tag{11.58}$$

(b) *What is the optimal class size when $w = 60$ and $p = 2$?*

Answer: Plugging these into the expression we just derived, we get an optimal class size of 25.6 students per class.

(c) *What happens to class size as teacher salaries change?*

Answer: We can simply take the derivative of our optimal class size expression with respect to w to see if it is positive or negative. Since it is clearly positive, we know that class size increases as teacher salaries increase (and decreases as teacher salaries decrease).

(d) *What happens to class size as the community's marginal willingness to pay for student achievement changes?*

Answer: This time we would take the derivative of our optimal class size expression with respect to p. Since this derivative is clearly negative, we know that class size will increase as p decreases (and decrease as p increases).

(e) *What would change if the state government subsidizes the local contribution to school spending?*

Answer: This would effectively lower w — and thus decrease class size. Alternatively you could think of it as raising the local willingness to spend money on schools — which would also have the effect of decreasing class size.

(f) *Now suppose that the community's marginal willingness to pay for additional student achievement is a function of the achievement level. In particular, suppose that $p(a) = Ba^{\beta-1}$ where $\beta \leq 1$. For what values of β and B is the problem identical to the one you just solved?*

Answer: If $B = p$ and $\beta = 1$, the problem is identical to what we solved before.

(g) *Solve for the optimal t given the marginal willingness to pay of $p(a)$. What is the optimal class size when $B = 3$ and $\beta = 0.95$ (assuming again that $w = 60$.*

Answer: We would now solve the problem

$$\max_{a,t} p(a)a - wt = Ba^{\beta-1}a - wt = Ba^{\beta} - wt \text{ subject to } a = 100t^{0.75} \tag{11.59}$$

which can be written as the unconstrained maximization problem

$$\max_{t} B\left(100t^{0.75}\right)^{\beta} - wt. \tag{11.60}$$

The first order condition for this problem is

$$0.75\beta B(100^{\beta})t^{0.75\beta-1} - w = 0, \tag{11.61}$$

which solves to

$$t = \left(\frac{w}{0.75\beta B\left(100^{\beta}\right)}\right)^{1/(0.75\beta-1)} \tag{11.62}$$

Substituting in $w = 60$, $\beta = 0.95$ and $B = 3$, we then get $t \approx 37.27$. Given that class size is $1000/t$, we get that the optimal class size under these conditions is approximately 26.83 students per class.

(h) *Under the parameter values just specified, does class size respond to changes in teacher salaries as it did before?*

Answer: Under the parameter values specified above, the optimal input level t becomes

$$t = \left(\frac{w}{0.75(0.95)(3)\left(100^{0.95}\right)}\right)^{1/(0.75(0.95)-1)} \approx 169.79w^{-3.48}. \tag{11.63}$$

This implies an optimal class size of

$$\text{Class Size} = \frac{1000}{t} = \frac{1000}{169.79w^{-3.48}} \approx 5.89w^{3.48}. \tag{11.64}$$

The derivative of class size with respect to w is then clearly positive — which implies that class size increases as teacher salaries increase.

Conclusion: Potentially Helpful Reminders

1. Remember that the one-step profit maximization method is completely equivalent to the two-step method that minimizes costs first. The former gives us the profit maximizing condition that $MRP = w$; the latter gives us the condition that $p = MC$. Since the two methods are equivalent, the two conditions imply one another in the single-input model; i.e. whenever $MRP = w$, it must be that $p = MC$ and vice versa if we are at a true optimum for the firm. (This insight will generalize to multiple inputs in the next chapter.)

2. Make sure you really understand how the 2-step method for maximizing profit is equivalent to simply maximizing profit in a single step. As the model becomes more complex when we allow for two (rather than one) input in the next chapter, we will increasingly rely on the 2-step method.

3. In the 2-step method, we begin by deriving the cost curve (or function) — and this involves no consideration of the output price p. Thus, any producer who hires inputs in competitive input markets will get the same cost curves for a given technology. In particular, it does not matter for the cost curves whether the producer is in fact a price taker in the output market — he could be a monopolist who sets price in the output market, but his cost curves would still arise in exactly the same way. This will become important in Chapters 23 and beyond where we will use the same types of cost curves to discuss markets that are not perfectly competitive (where firms are not price takers).

4. It's a good time to get used to the way we use the term *cost*. Whenever we say "cost" in this chapter and throughout this text, we mean "economic" or "opportunity" cost. If something has to be paid regardless of what action I choose, it is not a cost of choosing an action. Ever. (In Chapter 13 we will adopt the term "expense" for cases where we have to spend money regardless of what we do.)