CHAPTER

12

Production with Multiple Inputs

This chapter continues the treatment of producer theory when firms are price takers. Chapter 11 focused on the short run model in which capital is held fixed and labor is therefore the only variable input. This allowed us to introduce the ideas of profit maximization and cost minimization within the simplest possible setting. Chapter 12 now focuses on the long run model in which both capital and labor are variable. The introduction of a second input then introduces the possibility that firms will substitute between capital and labor as input prices change. It also introduces the idea of returns to scale. And we will see that the 2-step profit maximization approach that was introduced at the end of Chapter 11 — i.e. the approach that begins with costs and then adds revenues to the analysis — is much more suited to a graphical treatment than the 1-step profit maximization approach (which would require graphing in 3 dimensions.)

Chapter Highlights

The main points of the chapter are:

1. Profit maximization in the 2-input (long run) model is conceptually the same as it is for the one-input (short run) model — the profit maximizing production plans (that involve positive levels of output) again satisfying the condition that the **marginal revenue products of inputs are equal to the input prices**. The **marginal product** of each input is measured along the vertical slice of the production frontier that holds the *other* input fixed (as already developed for the marginal product of labor in Chapter 11.)

2. **Isoquants** are horizontal slices of the production frontier and are, in a technical sense, similar to indifference curves from consumer theory. Their shape indicates the degree of substitutability between capital and labor, and their slope is the (marginal) **technical rate of substitution** which is equal to the (negative) ratio of the marginal products of the inputs.

3. Unlike in consumer theory where the labeling of indifference curves had no cardinal meaning, the labeling on isoquants has a clear cardinal interpretation since production units are objectively measurable. The rate at which this labeling increases tells us whether the production frontier's slope is increasing at an increasing or decreasing rate — and thus whether the production technology is exhibiting increasing or decreasing **returns to scale**.

4. Cost minimization in the two-input model is considerably more complex than it was in the single-input model of Chapter 11 because there are now many different ways of producing any given output level without wasting inputs (i.e. in a technologically efficient way) as indicated by all input bundles on each isoquant. The **least cost way of producing** any output level then depends on input prices — and is graphically seen as the tangency between **isocosts** and isoquants.

5. For **homothetic production processes**, all cost minimizing input bundles will lie on the same ray from the origin within the isoquant graph. The vertical slice of the 3-D production frontier along that ray is then the relevant slice on which the profit maximizing production plan lies.

6. The **cost curve** is derived from the cost-minimizing input bundles on that same ray from the origin — and, analogous to what we did in Chapter 11, its shape is the inverse of the shape of the production frontier along that slice. (This shape also indicates whether the production process has increasing or decreasing returns to scale). Once we have derived the cost curve, the 2-step profit maximization proceeds exactly as it did in Chapter 11 — with output occurring where $p = MC$.

Using the LiveGraphs

For an overview of what is contained on the LiveGraphs site for each of the chapters (from Chapter 2 through 29) and how you might utilize this resource, see pages 2-3 of Chapter 1 of this *Study Guide*. To access the LiveGraphs for Chapter 12, click the *Chapter 12* tab on the left side of the LiveGraphs web site.

In addition to the *Animated Graphics*, the *Static Graphics* and the *Downloads* that accompany each of the graphs in the text of this chapter, we have some exciting **Exploring Relationships** modules for this chapter. In particular, the modules illustrate four types of production frontiers (or production functions) — and then slice these functions in three different ways:

1. *Horizontally* — giving rise to isoquants (that haver similarities to indifference curves from consumer theory).

2. *Vertically*, holding one of the inputs fixed — giving rise to single-input production frontiers like those we worked with in Chapter 11. The slopes of

these are equal to marginal product of labor (when capital is held fixed) and marginal product of capital (when labor is held fixed).

3. *Vertically*, along rays from the origin — giving rise to the slices along which cost minimizing bundles lie when the production technology is homothetic. This slice also illustrates whether the production process has decreasing, constant or increasing returns to scale.

One of the more interesting aspects of these modules lies in their ability to demonstrate how production frontiers can have *both diminishing marginal product of all inputs — and increasing returns to scale.* This is often a very difficult idea to wrap one's mind around — but it's easily illustrated mathematically. My hope is that with these graphical modules, we can make what's easy to see mathematically a bit easier to see intuitively.

12A Solutions to Within-Chapter-Exercises for Part A

Exercise 12A.1 *Suppose we are modeling all non-labor investments as capital. Is the rental rate any different depending on whether the firm uses money it already has or chooses to borrow money to make its investments?*

Answer: No — for the same reason that the rental rate of photocopiers for Kinkos is the same regardless of whether Kinkos owns or rents the copiers. If the firm borrows money from another firm, it is doing so at the interest rate r which then becomes the rental rate for the financial capital it is investing. If the firm uses its own money, it is foregoing the option of lending that money to another firm at the interest rate r — and thus it again costs the firm r per dollar to invest in its own capital.

Exercise 12A.2 *Explain why the vertical intercept on a three dimensional isoprofit plane is π/p (where π represents the profit associated with that isoprofit plane).*

Answer: A production plan on the vertical intercept has positive x but zero ℓ and k. Profit for a production plan (ℓ, k, x) is given by $\pi = px - w\ell - rk$ — but since $\ell = k = 0$ on the vertical axis, this reduces to $\pi = px$. Put differently, when there are no input costs, profit is the same as revenue for the firm — and revenue is just price times output. Dividing both sides of $\pi = px$ by p, we get π/p — the value of the intercept of the isoprofit plane associated with profit π.

Exercise 12A.3 *We have just concluded that $MP_k = r/p$ at the profit maximizing bundle. Another way to write this is that the marginal revenue product of capital $MRP_k = pMP_k$ is equal to the rental rate. Can you explain intuitively why this makes sense?*

Answer: The intuition is exactly identical to the intuition developed in Chapter 11 for the condition that marginal revenue product of labor must be equal to wage at the optimum. The marginal product of capital is the additional output we get from one more unit of capital (holding fixed all other inputs). Price times the marginal product of capital is the additional *revenue* we get from one more unit of capital. Suppose we stop hiring capital when the cost of a unit of capital r is exactly equal to this marginal revenue product of capital. Since marginal product is diminishing, this means that the marginal revenue from the previous unit of capital was greater than r — and so I made money on hiring the previous unit of capital. But if I hire past the point where $MRP_k = r$, I am hiring additional units of capital for which the marginal revenue is less than what it costs me to hire those units. Thus, had I stopped hiring before $MRP_k = r$, I would have forgone the opportunity of making additional profit from hiring more capital; if, on the other hand, I hire beyond $MRP_k = r$, I am incurring losses on the additional units of capital.

Exercise 12A.4 *Suppose capital is fixed in the short run but not in the long run.* True or False*: If the firm has its long run optimal level of capital k^D (in panel (f) of Graph 12.1), then it will choose ℓ^D labor in the short run. And if ℓ^A in panel (c) is not equal to ℓ^D in panel (f), it must mean that the firm does not have the long run optimal level of capital as it is making its short run labor input decision.*

Answer: This is true. If the firm has capital k^D, then it is operating on the short-run slice that holds k^D fixed in panel (f). The short run isoprofit is then just a slice of the long run isoprofit plane — and is tangent at labor input level ℓ^D. If the firm chooses $\ell^A \neq \ell^D$ in the short run, then it is not operating on this slice — and thus does not have the long run profit maximizing capital level of k^D.

Exercise 12A.5 *Apply the definition of an isoquant to the one-input producer model. What does the isoquant look like there? (*Hint*: Each isoquant is typically a single point.)*

Answer: An isoquant for a given level of output x is the set of all input bundles that result in that level of output without wasting any input. In the one-input model, the only production plans that don't waste inputs are those that lie on the production frontier. For each level of x, we therefore have a single level of (labor) input that can produce that level of x without any input being wasted. This single labor input level is then the isoquant for producing a particular output level x.

Exercise 12A.6 *Why do you think we have emphasized the concept of marginal product of an input in producer theory but not the analogous concept of marginal utility of a consumption good in consumer theory?*

Answer: The marginal product of an input is the number of additional units of output that can be produced if one more unit of the input is hired. This is an objectively measurable quantity. The marginal utility of a consumption good is the additional utility that will result from consumption of one more unit of the consumption good. Since it is measured in utility terms, it is not objectively measurable (since we have no way to measure "utils" objectively).

Exercise 12A.7 *Repeat this reasoning for the case where $MP_\ell = 2$ and $MP_k = 3$.*

Answer: Suppose we currently produce some quantity x using ℓ units of labor and k units of capital. If $MP_\ell = 2$ and $MP_k = 3$, this implies that, at my current production plan, capital is 1.5 times as productive as labor. Suppose I want to use one less unit of capital but continue to produce the same amount as before. Then, since capital is 1.5 times as productive as labor, this would imply I would have to hire 1.5 units of labor. In other words, substituting 1 unit of capital for 1.5 units of labor leads to no change in output on the margin — which is another way of saying that my technical rate of substitution is currently $TRS = -1/1.5 = -2/3$ — which is just $-MP_\ell/MP_k$.

Exercise 12A.8 *Is there a relationship analogous to equation (12.3) that exists in consumer theory and, if so why do you think we did not highlight it in our development of consumer theory?*

Answer: Yes. In exactly the same way, we could derive the relationship

$$MRS = -\frac{MU_1}{MU_2} \qquad (12.1)$$

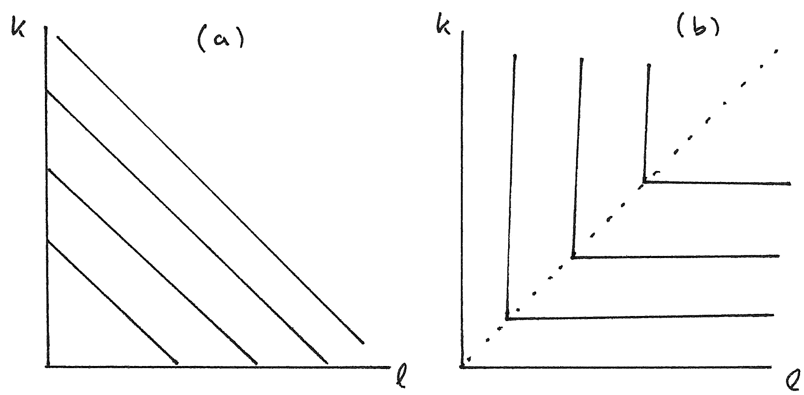
where MU_1 and MU_2 are the marginal utility of consuming good 1 and good 2. Since marginal utility is not objectively measurable, we did not emphasize the concept. However, note that "utils" cancel on the right hand side of our equation — implying that MRS is not expressed in util terms. Thus, MRS is a meaningful and measurable concept even if MU is not.

Exercise 12A.9 *In the "old days", professors used to hand-write their academic papers and then have secretaries type them up. Once the handwritten scribbles were handed to the secretaries, there were two inputs into the production process: secretaries (labor) and typewriters (capital). If one of the production processes in Graph 12.4 represents the production for academic papers, which would it be?*

Answer: There is little substitutability between secretaries and typewriters since each secretary has to be matched with one typewriter if papers are to be typed. Thus, panel (c) would come closest to representing the production for academic papers.

Exercise 12A.10 *What would isoquant maps with no substitutability and perfect substitutability between inputs look like? Why are they homothetic?*

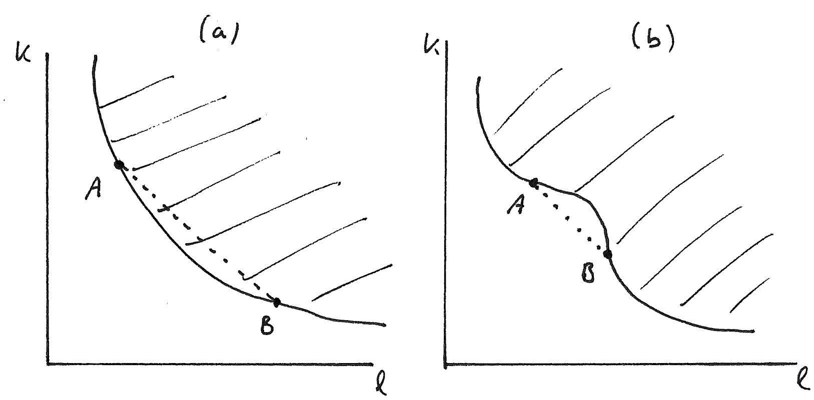
Answer: These are graphed in Graph 12.1, with panel (a) representing a production process with perfect substitutability of capital and labor and panel (b) representing perfect complementarity. These are both homothetic because the slope of the isoquants is unchanged along any ray from the origin.



Graph 12.1: Perfect Substitutes and Complements in Production

Exercise 12A.11 *Illustrate the upper contour set for an isoquant that satisfies our notion of "averages being better than extremes". Is it convex?*

Answer: This is illustrated in panel (a) of Graph 12.2 where the line connecting any point *A* and point *B* that lie within the shaded upper contour set also lies within that shaded set.



Graph 12.2: Upper Contour Sets

Exercise 12A.12 *Illustrate the upper contour set for an isoquant that does* not *satisfy our notion of "averages being better than extremes. Is it convex?*

Answer: This is illustrated in panel (b) of Graph 12.2 (previous page) where we can find a point A and another point B that lie within the shaded upper contour set but where the line connecting the two does not lie within that shaded set.

Exercise 12A.13 *Consider again a real-world mountain and suppose that the shape of any horizontal slice of this mountain is a perfect (filled in) circle. I have climbed the mountain from every direction — and I have found that the climb typically starts off easy but gets harder and harder as I approach the top because the mountain gets increasingly steep. Does this mountain satisfy any of the two notions of convexity we have discussed?*

Answer: A perfect (filled in) circle is a convex set. Thus, the horizontal slices of our mountain are convex sets — which means the mountain satisfies our original notion of convexity. If, however, the mountain gets steeper as we move up, vertical slices of the mountain will not be convex. Thus, our second notion of convexity does not hold.

Exercise 12A.14 True or False*: Convexity in the sense of "averages are better than extremes" is a necessary but not sufficient condition for convexity of the producer choice set.*

Answer: This is true. If convexity in the sense of "averages are better than extremes" does not hold, it means that horizontal slices of the producer choice set are non-convex — which makes the producer choice set non-conves. Thus, in order for the producer choice set to be convex, it must necessarily be true that the horizontal slices are convex. It is not, however, sufficient — because the producer choice set could have non-convex vertical slices even if all the horizontal slices are convex.

Exercise 12A.15 *Consider a single input production process with increasing marginal product. Is this production process increasing returns to scale? What about the production process in Graph 11.10?*

Answer: Increasing marginal product in the single input model implies that we can increase the input by a factor t and thereby will raise output by a factor greater than t. Thus, the production process is increasing returns to scale. In Graph 11.10, the production process has this feature initially — but eventually becomes decreasing returns to scale.

Exercise 12A.16 True or False: *For homothetic production frontiers, convexity of the producer choice set implies decreasing returns to scale.*

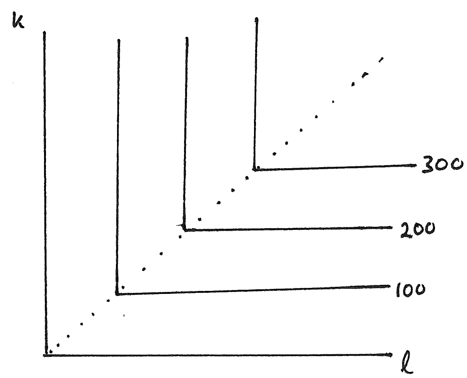
Answer: This is true. Convexity of the producer choice set implies that, as we "climb" the production frontier, the slope becomes shallower and shallower along any ray from the origin. This means that a t-fold increase in all inputs leads to less than a t-fold increase in production.

Exercise 12A.17 *If the three panels of Graph 12.6 represented indifference curves for consumers, would there be any meaningful distinction between them? Can you see why the concept of "returns to scale" is not meaningful in consumer theory?*

Answer: The distinction would not be meaningful — because the shape of the indifference curves and the ordering of the labels is the same in all three panels. Returns to scale is not meaningful in consumer theory because the statement "as I double the consumption bundle, my utility doubles" is not meaningful when we don't think we can measure utility objectively.

Exercise 12A.18 *On a graph with labor hours on the horizontal and computer hours on the vertical axis, illustrate the isoquants for 100, 200 and 300 typed pages of manuscript.*

Answer: This is illustrated in Graph 12.3.



Graph 12.3: Production of Academic Papers

Exercise 12A.19 *Suppose there are some gains to specialization in typing manuscripts — with some office assistants specializing in typing mathematical equations, others in typing text, yet others in incorporating graphics. Then, although labor and capital might remain perfect complements in production, the production process becomes increasing rather than constant returns to scale. Could you have diminishing marginal product in both inputs and still increasing returns to scale in production?*

Answer: Yes. The only thing that would change is that we would have the labels associated with isoquants in Graph 12.3 increase faster — but, for a fixed number of hours of computer time, we would still have the marginal product of labor decrease suddenly as we run out of computer time for the labor to use. Similarly, for a fixed number of hours of labor, the marginal product of additional computer time would fall once the labor time is all used up.

Exercise 12A.20 True or False*: In a two-input model, if marginal product is increasing for one of the inputs, then the production process has increasing returns to scale.*

Answer: This is true. Suppose the production process has increasing marginal product of labor. This means that, *holding capital fixed*, the output from each additional unit of labor will be larger than from the previous. Thus, a t-fold increase in labor results in more than a t-fold increase in output. If this is true *when capital stays constant*, then it must certainly also be true when capital is simultaneously increased t-fold. Thus, as all inputs are increased t-fold, output increases more than t-fold — i.e. we have increasing returns to scale.

Exercise 12A.21 True or False*: In the sinlge-input model, each isoquant is composed of a single point which implies that all technologically efficient production plans are also economically efficient.*

Answer: This is true. An isoquant is the set of input bundles that can produce a given output level without inputs being wasted. Since there is only one input, there is only one way to produce each output level without wasting inputs — thus the isoquant is a single point. It is technologically efficient because no input is wasted — and economically efficient because it is (by default) the least expensive of all the technologically efficient input bundles.

Exercise 12A.22 True or False*: In the two input model, every economically efficient production plan must be technologically efficient but not every technologically efficient production plan is necessarily economically efficient.*

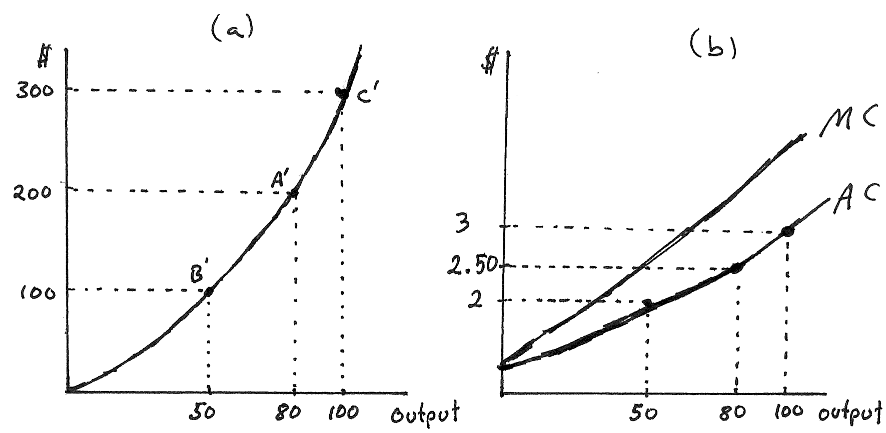
Answer: This is true. In order for an input bundle to be the economically most efficient — or cheapest — way of producing an output level, it must be the case that no inputs are wastes — i.e. the input bundle must be technologically efficient for this output level. But, when there are many technologically efficient ways of producing a given level of output, some will be more expensive and some less — so they cannot all be economically efficient (i.e. cheapest).

Exercise 12A.23 True or False*: We have to know nothing about prices, wages or rental rates to determine the technologically efficient ways of producing different output levels, but we cannot generally find the economically efficient ways of producing any output level without knowing these.*

Answer: This is true. Technologically efficient production just means production without wasting inputs — and we do not have to know anything about prices in the economy to know whether we are wasting inputs. Put differently, we do not have to know anything about prices to derive isoquants — they just come from the production frontier which is determined by the technology that is available to the producer. Economically efficient production means the “cheapest” way to produce — and that of course has much to do with input prices. (It does not, of course, have anything to do with the output price.)

Exercise 12A.24 *Suppose the numbers associated with the isoquants in Graphs 17.7(a) and (b) had been 50, 80 and 100 instead of 50, 100 and 150. What would the total cost, MC and AC curves look like? Would this be an increasing or decreasing returns to scale production process, and how does this relate to the shape of the cost curves?*

Answer: This is illustrated in Graph 12.4. This would imply it is getting increasingly hard to produce additional units of output — i.e. the underlying technology represented by the isoquants has decreasing returns to scale. As a result, the cost of producing is increasing at an increasing rate — which causes the MC and AC curves to slope up.



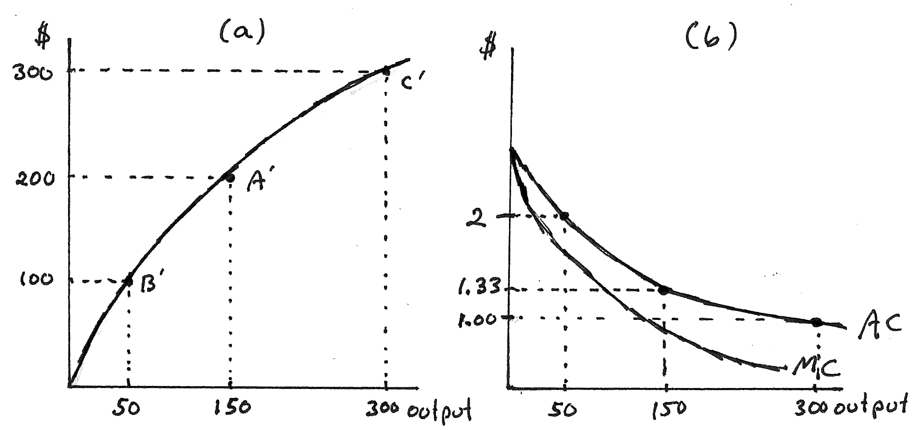
Graph 12.4: Decreasing Returns to Scale Cost Curves

Exercise 12A.25 *How would your answer to the previous question change if the numbers associated with the isoquants were 50, 150, and 300 instead?*

Answer: This is illustrated in Graph 12.5 (next page). Production of additional goods is getting increasingly easy — which means the underlying production technology has increasing returns to scale. As a result, increased production causes costs to increase at a decreasing rate — which implies MC and AC are downward sloping.

Exercise 12A.26 *If w increases, will the economically efficient production plans lie on a steeper or shallower ray from the origin? What if r increases?*

Answer: If w increases, then w/r increases — which means the slope of the isocosts becomes steeper. Thus, the tangencies with isoquants will occur to the left (where the isoquants are steeper) — implying that they will occur on a ray that is



Graph 12.5: Increasing Returns to Scale Cost Curves

steeper. If r increases, w/r falls — meaning that the isocosts get shallower. Thus, the tangencies with isoquants will occur to the right (where the isoquants are shallower) — implying that they will occur on a ray that is shallower. This should make sense — as w increases, economic efficiency will require a substitution away from labor and toward more capital, and the reverse will happen if r increases.

Exercise 12A.27 *What is the shape of such a production process in the single input case? How does this compare to the shape of the vertical slice of the 3-dimensional production frontier along the ray from the origin in our graph?*

Answer: The shape of such a one-input production process is the usual shape we employed in Chapter 11: On a graph with labor on the horizontal and output on the vertical, the production frontier initially increases at an increasing rate (as it becomes easier and easier to produce additional output) but eventually increases at a decreasing rate (as it becomes increasingly hard to produce additional output.) This is exactly the same shape as the slice along a ray from the origin of the 2-input production process that has initially increasing and eventual decreasing returns to scale.

Exercise 12A.28 True or False*: If a producer minimizes costs, she does not necessarily maximize profits, but if she maximizes profits, she also minimizes costs. (*Hint*: Every point on the cost curve is derived from a producer minimizing the cost of producing a certain output level.)*

Answer: True. Any production plan that is represented along the cost curve is cost minimizing, but only the plan where $p = MC$ is profit maximizing. But since the profit maximizing point is derived from the cost curve, it implicitly is also cost minimizing.

Exercise 12A.29 *Suppose a production process begins initially with increasing returns to scale, eventually assumes constant returns to scale but never has decreasing returns. Would the MC curve ever cross the AC curve?*

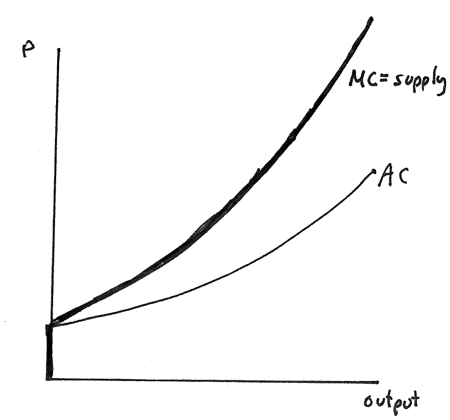
Answer: No, it would never cross AC. The MC and AC curves would start at the same place, with MC falling faster than AC along the increasing returns to scale portion of production. When we reach the constant returns to scale portion, MC would become flat, and AC would continue to fall at a decreasing rate as it converges (but never quite reaches) the flat MC curve.

Exercise 12A.30 *Another special case is the one graphed in Graph 12.7. What are the optimal supply choices for such a producer as the output price changes?*

Answer: When $p^* = MC$, any output quantity would be optimal; when $p^* < MC$, it is optimal to produce zero (since profit would be negative); when $p^* > MC$, it would be optimal to produce an infinite amount (since you can keep making profit on each additional unit produced. Thus, the supply curve would lie on the vertical axis between $p = 0$ and $p = p^*$, horizontal at p^* and "vertical at infinity" for $p > p^*$.

Exercise 12A.31 *Illustrate the output supply curve for a producer whose production frontier has decreasing returns to scale throughout (such as the case illustrated in Graph 12.1).*

Answer: This is illustrated in Graph 12.6. Decreasing returns to scale lead to a MC curve that is increasing throughout. Since it begins where AC begins, the entire MC curve lies above AC — and thus the entire MC curve is the supply curve.



Graph 12.6: Supply Curve with Decreasing Returns to Scale

12B Solutions to Within-Chapter-Exercises for Part B

Exercise 12B.1 *Just as we can take the partial derivative of a production function with respect to one of the inputs (and call it the "marginal product of the input"), we could take the partial derivative of a utility function with respect to one of the consumption goods (and call it the "marginal utility from that good"). Why is the first of these concepts economically meaningful but the second is not?*

Answer: This is because utility is not objectively measurable whereas output is. It is therefore meaningful to ask "how much additional output will one more unit of labor produce", but it is not meaningful to ask "how much additional utility will one more unit of good x yield."

Exercise 12B.2 *Using the same method employed to derive the formula for MRS from a utility function, derive the formula for TRS from a production function $f(\ell, k)$.*

Answer: The technical rate of substitution (TRS) is simply the change in k divided by the change in ℓ such that output remains unchanged, or

$$\frac{\Delta k}{\Delta \ell} \text{ such that } \Delta x = 0. \tag{12.2}$$

Actually, what we mean by a technical rate of substitution is somewhat more precise — we are not looking for just *any* combination of changes in k and ℓ (such that Δx=0). Rather, we are looking for small changes that define the slope around a particular point. Such small changes are denoted in calculus by using "d" instead of "Δ". Thus, we can re-write (12.2) as

$$\frac{dk}{d\ell} \text{ such that } dx = 0. \tag{12.3}$$

Changes in output arise from the combined change in k and ℓ, and this is expressed as the total differential (dx)

$$dx = \frac{\partial f}{\partial \ell} d\ell + \frac{\partial f}{\partial k} dk. \tag{12.4}$$

Since we are interested in changes in input bundles that result in no change in output (thus leaving us on the same isoquant), we can set expression (12.4) to zero

$$\frac{\partial f}{\partial \ell} d\ell + \frac{\partial f}{\partial k} dk = 0 \tag{12.5}$$

and then solve out for $dk/d\ell$ to get

$$\frac{dk}{d\ell} = -\frac{(\partial f/\partial \ell)}{(\partial f/\partial k)}. \tag{12.6}$$

Since this expression for $dk/d\ell$ was derived from the expression $dx = 0$, it gives us the equation for small changes in k divided by small changes in ℓ such that production remains unchanged — which is precisely our definition of a technical rate substitution.

Exercise 12B.3 True or False*: Producer choice sets whose frontiers are characterized by quasiconcave functions have the following property: All horizontal slices of the choice sets are convex sets.*

Answer: This is true — the horizontal slices of the quasiconcave functions are isoquants that satisfy the "averages are better than extremes" property — which means the set of production plans that lie above the isoquant (and thus inside the producer choice set) is convex.

Exercise 12B.4 True or False*: All quasiconcave production functions — but not all concave production functions — give rise to convex producer choice sets.*

Answer: This is false. Since all concave production functions are also quasiconcave, whatever holds for quasiconcave production functions must hold for concave productions. The statement would be true of the terms "quasiconcave" and "concave" switched places.

Exercise 12B.5 True or False*: Both quasiconcave and concave production functions represent production processes for which the "averages are better than extremes" property holds.*

Answer: This is true. We have shown that quasiconcave production functions give rise to producer choice sets whose horizontal slices are convex sets — which in turn implies that the isoquants have the usual shape that satisfies "averages are better than extremes." And since all concave functions are also quasiconcave, the same must hold for concave production functions.

Exercise 12B.6 *Verify the last statement regarding Cobb-Douglas production functions.*

Answer: The Cobb-Douglas production function takes the form $f(\ell,k) = \ell^\alpha k^\beta$. When we multiply a given input bundle (ℓ,k) by some factor t, we get

$$f(t\ell,tk) = (t\ell)^\alpha (tk)^\beta = t^{(\alpha+\beta)}\ell^\alpha k^\beta = t^{(\alpha+\beta)} f(\ell,k). \tag{12.7}$$

When $\alpha+\beta = 1$, this equation tells us that increasing the inputs be a factor of t results in an increase of output by a factor of t — which is the definition of constant returns to scale. When $\alpha+\beta < 1$, the equation tells us that such an increase in inputs will result in less than a t-fold increase in output — which is the definition of decreasing returns to scale. And when $\alpha+\beta > 1$, output increases by more that t-fold — giving us increasing returns to scale.

Exercise 12B.7 *Can you give an example of a Cobb-Douglas production function that has increasing marginal product of capital and decreasing marginal product of labor? Does this production function have increasing, constant or decreasing returns to scale?*

Answer: In order for the example to work, the function $f(\ell,k) = \ell^{\alpha}k^{\beta}$ would have to be such that $\beta > 1$ (to get increasing marginal product of capital) and $\alpha < 1$ (to get decreasing marginal product of labor). Since we would still have $\alpha > 0$, this implies that $\alpha+\beta > 1$ — i.e. the production function has increasing returns to scale. This should make intuitive sense: If I can increase just *one* input t-fold and get a greater than t-fold increase in output (as I can if the marginal product of capital is increasing), then I can certainly increase *both* inputs t-fold and get more than a t-fold increase in output. So — as long as we have increasing marginal product in one input, we have increasing returns to scale.

Exercise 12B.8 True or False*: It is not possible for a Cobb-Douglas production process to have decreasing returns to scale and increasing marginal product of one of its inputs.*

Answer: This follows immediately from our answer to the previous exercise: Increasing marginal product in the Cobb-Douglas production function implies an exponent greater than 1 — but that implies a sum of exponents greater than 1 which is in turn equivalent to increasing returns to scale. Therefore the statement is true — you cannot have decreasing returns to scale and increasing marginal product at the same time.

Exercise 12B.9 *In a 3-dimensional graph with x on the vertical axis, can you use the equation (12.18) to determine the vertical intercept of an isoprofit curve $P(\pi, p, w, r)$? What about the slope when k is held fixed?*

Answer: At the vertical intercept, $k = \ell = 0$ — which implies the equation simply becomes $\pi = px$ or $x = \pi/p$ which is the intercept on the vertical (x) axis. When k is held fixed at, say, $\overline{k}$, The equation becomes $\pi = px - w\ell - r\overline{k}$. Rearranging terms, we can write this as

$$x = \left(\frac{\pi + r\overline{k}}{p}\right) + \frac{w}{p}\ell. \tag{12.8}$$

This is then an equation of the part of the production frontier that falls on the vertical slice that holds k fixed at $\overline{k}$. It has an intercept equal to the term in parenthesis, and it's slope is w/p.

Exercise 12B.10 *Define profit and isoprofit curves for the case where land L is a third input and can be rented at a price r_L.*

Answer: Profit is then simply

$$\pi = px - w\ell - rk - r_L L, \tag{12.9}$$

and the isoprofit plane P is

$$P(\pi,p,w,r,r_L)=\left\{(x,\ell,k,L)\in\mathbb{R}^4\mid \pi=px-w\ell-rk-r_LL\right\}. \tag{12.10}$$

Exercise 12B.11 *Demonstrate that the problem as written in (12.20) gives the same answer.*

Answer: Setting up the Lagrange function for this problem gives

$$\mathcal{L}(x,\ell,k,\lambda)=px-w\ell-rk+\lambda(x-f(\ell,k)), \tag{12.11}$$

which results in first order conditions

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x}&=p+\lambda=0,\\
\frac{\partial\mathcal{L}}{\partial \ell}&=-w-\lambda\frac{\partial f(\ell,k)}{\partial\ell}=0,\\
\frac{\partial\mathcal{L}}{\partial k}&=-r-\lambda\frac{\partial f(\ell,k)}{\partial k}=0,\\
\frac{\partial\mathcal{L}}{\partial \lambda}&=x-f(\ell,k)=0.
\end{aligned} \tag{12.12}$$

Solving the first of these equations for $\lambda=-p$, substituting this into the second and third equations and rearranging terms then gives

$$w=p\frac{\partial f(\ell,k)}{\partial\ell} \text{ and } r=p\frac{\partial f(\ell,k)}{\partial k}, \tag{12.13}$$

which can further be written as

$$w=pMP_\ell=MRP_\ell \text{ and } r=pMP_k=MRP_k. \tag{12.14}$$

Exercise 12B.12 *Demonstrate that solving the problem as defined in equation (12.27) results in the same solution.*

Answer: The Lagrange function for this problem is

$$\mathcal{L}(x,\ell,k,\lambda)=px-w\ell-rk+\lambda(x-20\ell^{2/5}k^{2/5}). \tag{12.15}$$

The first order conditions for this problem are

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x}&=p+\lambda=0,\\
\frac{\partial\mathcal{L}}{\partial \ell}&=-w-8\lambda\ell^{-3/5}k^{2/5}=0,\\
\frac{\partial\mathcal{L}}{\partial k}&=-r-8\lambda\ell^{2/5}k^{-3/5}=0,\\
\frac{\partial\mathcal{L}}{\partial \lambda}&=x-20\ell^{2/5}k^{2/5}=0.
\end{aligned} \tag{12.16}$$

Plugging the $\lambda = -p$ (derived from the first equation) into the second and third equations then gives the condition that input prices are equal to marginal revenue products:

$$w = 8p\ell^{-3/5}k^{2/5} \text{ and } r = 8p\ell^{2/5}k^{-3/5}. \tag{12.17}$$

From this point forward, the problem solves out exactly as in the text. Solving the second of the two equations for k and plugging it into the first, we get the labor demand function

$$\ell(p,w,r) = \frac{(8p)^5}{r^2 w^3}, \tag{12.18}$$

and plugging this in for ℓ in the second equation, we get the capital demand function

$$k(p,w,r) = \frac{(8p)^5}{w^2 r^3}. \tag{12.19}$$

Finally, we can derive the output supply function by plugging equations (12.18) and (12.19) into the production function $f(\ell,k) = 20\ell^{2/5}k^{2/5}$ to get

$$x(p,w,r) = 20\frac{(8p)^4}{(wr)^2} = 81920\frac{p^4}{(wr)^2}. \tag{12.20}$$

Exercise 12B.13 *Each panel of Graph 12.12 illustrates one of three "slices" of the respective function through the production plan ($x = 1280, \ell = 128, k = 256$). What are the other two slices for each of the three functions? Do they slope up or down?*

Answer: For the supply function, the other two slices are

$$\begin{aligned} x(5,w,10) &= 81920\frac{5^4}{(10w)^2} = \frac{512000}{w^2} \text{ and} \\ x(5,20,r) &= 81920\frac{5^4}{(20r)^2} = \frac{128000}{r^2}, \end{aligned} \tag{12.21}$$

both of which slope down. This makes sense: As input prices increase, less output is produced. For the labor demand function, the two other slices are

$$\ell(p,20,10) = \frac{(8p)^5}{(10^2)(20^3)} \approx 0.0401p^5 \text{ and } \ell(5,20,r) = \frac{(8(5))^5}{20^3 r^2} = \frac{12800}{r^2}. \tag{12.22}$$

The slope is positive for the first and negative for the second. Thus, labor demand increases as output price increases but decreases as the rental rate of capital increases.

Finally, for the capital demand function, the other two slices are

$$k(p,20,10) = \frac{(8p)^5}{20^2(10^3)} \approx 0.082p^5 \quad \text{and} \quad k(5,w,10) = \frac{(8(5))^5}{10^3 w^2} = \frac{102400}{w^2}. \tag{12.23}$$

Again, the slope is positive for the first and negative for the second of these. Thus, capital demand increases as output price increases but decreases as wage increases.

Exercise 12B.14 *Did we calculate a "conditional labor demand" function when we did cost minimization in the one-input model?*

Answer: Yes, but we did not have to solve a "cost minimization" problem to do so. The only reason we need to solve a cost minimization problem now is that there are many technologically efficient production plans for each output level to choose from — and the problem allows us to determine which of these is the cheapest for a given set of input prices. In the one-input model, there was only one technologically efficient way of producing each output level — so we already knew that this was the cheapest way to produce. Thus, all we needed to do was invert the production function $x = f(\ell)$ — so that we could get the function $\ell(x)$ that told us how much labor input we needed to produce any output level. This function was then our "conditional labor demand" function — it told us, conditional on how much we want to produce, how much labor we will demand. In this case, input price was not part of the function because we knew that we would need that much labor to produce each output level no matter what the input price.

Exercise 12B.15 *Why are the conditional input demand functions not a function of output price p?*

Answer: Conditional input demands tell us least cost way of producing some output level x. The output price has no relevance for determining what the least cost way of producing is — it is only relevant for determining how much we should produce in order to maximize the difference between cost and revenue. Thus, only unconditional input demands are a function of output price.

Exercise 12B.16 *Suppose you are determined to produce a certain output quantity $\overline{x}$. If the wage rate goes up, how will your production plan change? What if the rental rate goes up?*

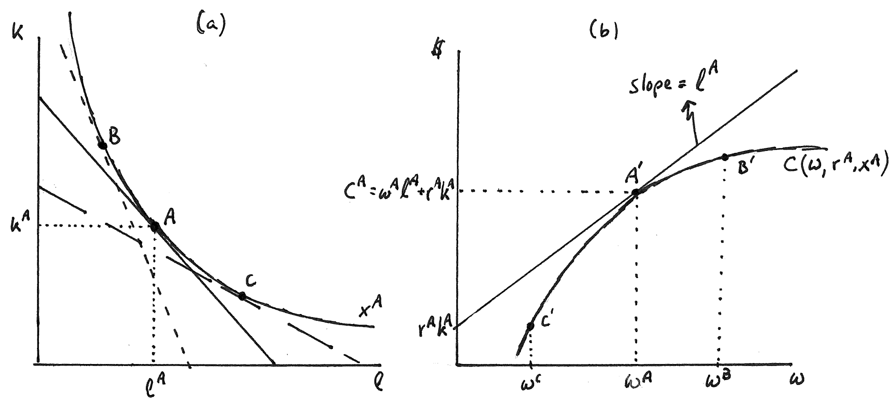
Answer: We can take the partial derivatives of the input demand functions with respect to wage to get

$$\frac{\partial \ell(w,r,\overline{x})}{\partial w} = \frac{-r^{1/2}}{2w^{3/2}}\left(\frac{x}{20}\right)^{5/4} < 0 \quad \text{and} \quad \frac{\partial k(w,r,\overline{x})}{\partial w} = \frac{1}{(wr)^{1/2}}\left(\frac{x}{20}\right)^{5/4} > 0. \tag{12.24}$$

Thus, when w increases, you will substitute away from labor and toward capital. The reverse holds if r increases (for similar reasons.)

Exercise 12B.17 *Can you replicate the graphical proof of the concavity of the expenditure function in the Appendix to Chapter 10 to prove that the cost function is concave in w and r?*

Answer: The relevant section in the Appendix to Chapter 10 begins with "Suppose that a consumer initially consumes a bundle A when prices of x_1 and x_2 are p_1^A and p_2^A, and suppose that the consumer attains utility level u^A as a result." Let's re-write this sentence to make it apply to the producer's cost minimization problem: "Suppose that a *producer* initially *employs* a bundle A when prices of ℓ and k are w^A and r^A, and suppose that the *producer produces an output* level x^A as a result." This input bundle A is graphed in panel (a) of Graph 12.7 where the slope of the (solid) isocost tangent to the x^A isoquant is $-w^A/r^A$.



Graph 12.7: Concavity of $C(w,r,x)$ in w

The lowest cost at which x^A can be produced when input prices are w^A and r^A is therefore $C(w^A,r^A,x^A) = C^A = w^A\ell^A + r^A k^A$. This is plotted in panel (b) of the graph where w is graphed on the horizontal and cost is graphed on the vertical axis. Since r^A and x^A are held fixed, we are in essence going to graph the slice of the cost function along which w varies. So far, we have plotted only one such point labeled A'.

Now suppose that w increases. If the producer does not respond by changing her input bundle, her cost will be given by the equation $C = r^A k^A + w\ell^A$ as w changes — and this is just the equation of a line with intercept $r^A k^A$ and slope ℓ^A. This line is plotted in panel (b) of the graph and represents the costs as w changes assuming the producer naively stuck with the same input bundle (ℓ^A, k^A). But of course the producer does not do this — because she can reduce her costs by substituting away from labor and toward more capital as she slides to the new cost-minimizing input bundle B that has the new (steeper) isocost tangent to the x^A isoquant. Thus, as w increases to w^B, her costs will go up by *less* than the naive linear cost line in panel (b) suggests. The same logic implies that the producer's costs will fall by more than what is indicated by the line if w falls to w^C. This results in the cost function slice $C(w,r^A,x^A)$ taking on the concave shape in the graph. Put differently, even if the producer never substituted toward inputs that have become relatively cheaper and away from inputs that have become relatively more expensive, this slice of the cost function would be a straight line (and thus "weakly" concave). Any ability to substitute between inputs then causes the strict concavity we have derived. The same logic applies to changes in r.

Exercise 12B.18 *What is the elasticity of substitution between capital and labor if the relationships in equation (12.51) hold with equality?*

Answer: If these relationships hold with equality, then this implies that a cost-minimizing producer will not change her input bundle to produce a given output level as input prices change. In other words, as some inputs become relatively cheaper and others relatively more expensive, the producer does not substitute away from the more expensive to the cheaper. This can only be cost-minimizing if in fact the technology is such that substituting between inputs is not possible — which is the same as saying that the elasticity of substitution is zero.

Exercise 12B.19 *Demonstrate how these indeed result from an application of the Envelope Theorem.*

Answer: Substituting the constraint into the objective, we can write the profit maximization problem in an unconstrained form; i.e.

$$\max_{\ell,k} \pi = pf(\ell,k) - w\ell - rk. \tag{12.25}$$

The "Lagrangian" is then simply equal to $\mathcal{L} = pf(\ell,k) - w\ell - rk$ (since there is no constraint to be multiplied by λ. The solution to the optimization problem is $\ell(w,r,p$ and $k(w,r,p)$. Substituting this solution into the objective function, we get the profit function $\pi(w,r,p)$ that tells us profit or any combination of prices (assuming the producer is profit maximizing). The envelope theorem then tells us that the derivative of this profit function with respect to a parameter (such as input and output prices) is equal to the derivative of the Lagrangian (which is just equal to the π expression in our optimization problem) with respect to that parameter *evaluated at the optimum* — i.e. evaluated at $\ell(w,r,p)$ and $k(w,r,p$. Thus,

$$\frac{\pi(w,r,p)}{\partial w} = \frac{\partial \mathcal{L}}{\partial w}\Big|_{\ell(w,r,p),k(w,r,p)} = -\ell\Big|_{\ell(w,r,p),k(w,r,p)} = -\ell(w,r,p), \tag{12.26}$$

and

$$\frac{\pi(w,r,p)}{\partial r} = \frac{\partial \mathcal{L}}{\partial r}\Big|_{\ell(w,r,p),k(w,r,p)} = -k\Big|_{\ell(w,r,p),k(w,r,p)} = -k(w,r,p). \tag{12.27}$$

Finally,

$$\frac{\pi(w,r,p)}{\partial r} = \frac{\partial \mathcal{L}}{\partial p}\Big|_{\ell(w,r,p),k(w,r,p)} = f(\ell,k)\Big|_{\ell(w,r,p),k(w,r,p)} \\ = f\left(\ell(w,r,p),k(w,r,p)\right) = x(w,r,p). \tag{12.28}$$

Exercise 12B.20 *How can you tell from panel (a) of the graph that* $\pi(x^B,\ell^B) > \pi' > \pi(x^A,\ell^A)$*?*

Answer: The intercept of the new (magenta) isoprofit is higher than the intercept of the original (blue) isoprofit. Let the new intercept be denoted π^B/p^B and the original intercept as π^A/p^A. We know that

$$\frac{\pi^B}{p^B} > \frac{\pi^A}{p^A} \text{ and } p^B > p^A, \tag{12.29}$$

which can be true only if $\pi^B > \pi^A$. Similarly,

$$\frac{\pi'}{p^B} > \frac{\pi^A}{p^A} \text{ and } p^B > p^A \text{ implies } \pi' > \pi^A. \tag{12.30}$$

Finally,

$$\frac{\pi'}{p^B} > \frac{\pi'}{p^B} \text{ implies } \pi^B > \pi'. \tag{12.31}$$

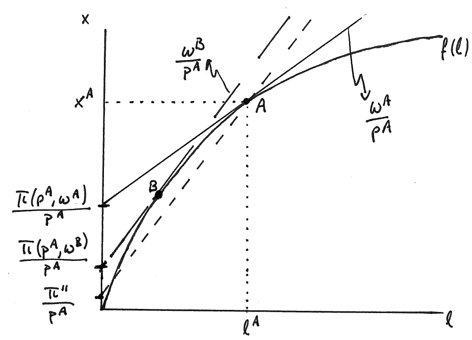
These three conclusions together imply $\pi^B > \pi' > \pi^A$.

Exercise 12B.21 *Use a graph similar to that in panel (a) of Graph 12.14 to motivate Graph 12.15.*

Answer: This is done in Graph 12.8 (next page) where the short run production function $f(\ell)$ is plotted with the originally optimal production plan (ℓ^A, x^A) at the original prices (w^A, p^A). An increase in the wage to w^B causes isoprofits to become steeper — with B becoming the new profit maximizing production plan. Had the producer not responded by changing her production plan, she would have operated on the steeper isoprofit that does through A rather than the one that goes through B — and would have made profit π'' instead of $\pi(p^A, w^B)$. Since the intercepts of the three isoprofits all have p^A in the denominator, it is immediate from the picture that

$$\pi(p^A, w^A) > \pi(p^A, w^B) > \pi'', \tag{12.32}$$

exactly as in the graph of the text.



Graph 12.8: Deriving the convexity of the profit function in w

12C Solutions to End-of-Chapter Exercises

Exercise 12.1

In our development of producer theory, we have found it convenient to assume that the production technology is homothetic.

A: *In each of the following, assume that the production technology you face is indeed homothetic. Suppose further that you currently face input prices (w^A, r^A) and output price p^A — and that, at these prices, your profit maximizing production plan is $A = (\ell^A, k^A, x^A)$.*

(a) *On a graph with ℓ on the horizontal and k on the vertical, illustrate an isoquant through the input bundle (ℓ^A, k^A). Indicate where all cost minimizing input bundles lie given the input prices (w^A, r^A).*

Answer: This is depicted in panel (a) of Graph 12.9 (next page). Since the isocosts must be tangent at the profit maximizing input bundle A, homotheticity implies that all tangencies of isocosts with isoquants lie on the ray from the origin that passes through A.

(b) *Can you tell from what you know whether the shape of the production frontier exhibits increasing or decreasing returns to scale along the ray you indicated in (a)?*

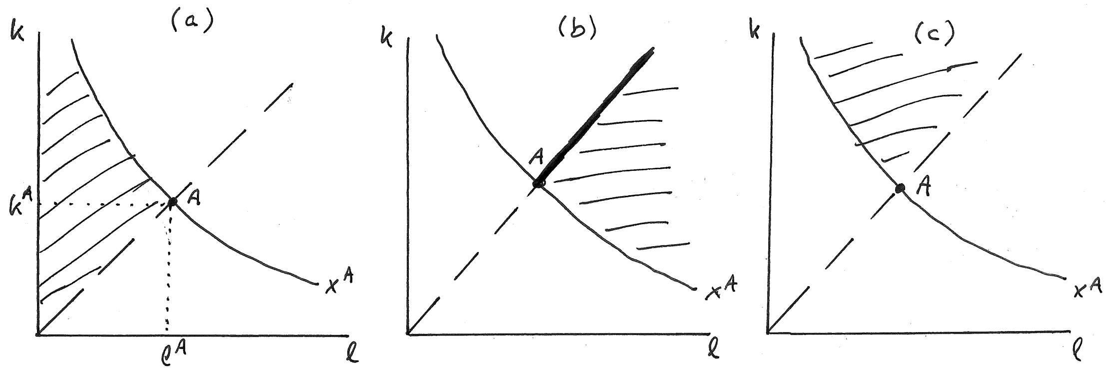
Answer: You cannot tell whether the production frontier has increasing or decreasing returns to scale along the entire ray from the origin.

(c) *Can you tell whether the production frontier has increasing or decreasing returns to scale around the production plan $A = (\ell^A, k^A, x^A)$?*

Answer: Yes, you can tell that it must have decreasing returns to scale at A — because the isoprofit must be tangent at that point in order for A to be the profit maximizing production plan.

(d) *Now suppose that wage increases to w'. Where will your new profit maximizing production plan lie relative to the ray you identified in (a)?*

Answer: When w increases, the isocosts become steeper — which implies that they are tangent to the isoquants to the left of the ray that goes through A. Thus, the new ray on which all cost minimizing production plans lie is steeper than the ray drawn in panel (a) of Graph 12.9. Since the new profit maximizing production plan must lie on that ray (because profit maximization implies cost minimization), the new profit maximizing production plan must lie to the left of the ray that passes through A.



Graph 12.9: Changing Prices and Profit Maximization

(e) *In light of the fact that supply curves shift to the left as input prices increase, where will your new profit maximizing input bundle lie relative to the isoquant for* x^A*?*

Answer: The leftward shift of supply curves as w increases implies that the profit maximizing output level falls. Thus, the new profit maximizing input bundle must lie *below* the x^A isoquant.

(f) *Combining your insights from (d) and (e), can you identify the region in which your new profit maximizing bundle will lie when wage increases to* w'*?*

Answer: This is illustrated as the shaded area in panel (a) of Graph 12.9. The shaded area emerges from the insight in (d) that the new profit maximizing bundle lies to the *left* of the ray through A and from the insight in (e) that it must lie *below* the isoquant for x^A.

(g) *How would your answer to (f) change if wage fell instead?*

Answer: If wage falls instead, then the isocosts become shallower — which implies that all cost minimizing bundles will now lie to the *right* of the ray through A. A drop in w will furthermore shift the output supply curve to the right — which implies that the profit maximizing production plan will involve an increase in the production of x. Thus, the new profit maximizing plan must lie to the *left* of the ray through A (because profit maximization implies cost minimization) and it must lie *above* the isoquant for x^A (because output increases). This is indicated as the shaded area in panel (b) of Graph 12.9.

(h) *Next, suppose that, instead of wage changing, the output price increases to* p'*. Where in your graph might your new profit maximizing production plan lie? What if* p *decreases?*

Answer: When output price p changes, the slopes of the isocosts (which are equal to $-w/r$) remain unchanged. Thus, all cost minimizing production plans remain on the ray through A. Since supply curves slope up, an increase in p will cause an increase in output — implying that the new profit maximizing production plan lies *above* the isoquant for x^A. Thus, when p increases, the new profit maximizing production plan lies on the bold portion of the ray through A as indicated in panel (b) of Graph 12.9. When p decreases, on the other hand, output falls — which implies that the new profit maximizing production plan lies on the dashed portion of the ray through A in panel (b) of the graph.

(i) *Can you identify the region in your graph where the new profit maximizing plan would lie if instead the rental rate* r *fell?*

Answer: If r falls, the isocosts become steeper — implying the ray containing all cost minimizing production plans will be steeper than the ray through A. Thus, cost minimization implies that the new profit maximizing input bundle will lie to the *left* of the ray through A.

A decrease in r further implies a shift in the supply curve to the right — which implies that output will increase. Thus, the profit maximizing input bundle must lie *above* the isoquant for x^A. This gives us the region to the *left* of the ray through A and *above* the isoquant x^A — which is equal to the shaded region in panel (c) of Graph 12.9.

B: *Consider the Cobb-Douglas production function $f(\ell,k) = A\ell^{\alpha}k^{\beta}$ with $\alpha,\beta > 0$ and $\alpha+\beta < 1$.*

(a) *Derive the demand functions $\ell(w,r,p)$ and $k(w,r,p)$ as well as the output supply function $x(w,r,p)$.*

Answer: These result from the profit maximization problem

$$\max_{\ell,k,x} px - w\ell - rk \text{ subject to } x = A\ell^{\alpha}k^{\beta} \tag{12.33}$$

which can also be written as

$$\max_{\ell,k} pA\ell^{\alpha}k^{\beta} - w\ell - rk. \tag{12.34}$$

Taking first order conditions and solving these, we then get input demand functions

$$\ell(w,r,p) = \left(\frac{pA\alpha^{(1-\beta)}\beta^{\beta}}{w^{(1-\beta)}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \text{ and } k(w,r,p) = \left(\frac{pA\alpha^{\alpha}\beta^{(1-\alpha)}}{w^{\alpha}r^{(1-\alpha)}}\right)^{1/(1-\alpha-\beta)}. \tag{12.35}$$

Plugging these into the production function and simplifying, we also get the output supply function

$$x(w,r,p) = \left(\frac{Ap^{(\alpha+\beta)}\alpha^{\alpha}\beta^{\beta}}{w^{\alpha}r^{\beta}}\right)^{1/(1-\alpha-\beta)} \tag{12.36}$$

(b) *Derive the conditional demand functions $\ell(w,r,x)$ and $k(w,r,x)$.*

Answer: We need to solve the cost minimization problem

$$\min_{\ell,k} w\ell + rk \text{ subject to } x = A\ell^{\alpha}k^{\beta}. \tag{12.37}$$

Setting up the Lagrangian and solving the first order conditions, we then get the conditional input demand functions

$$\ell(w,r,x) = \left(\frac{\alpha r}{\beta w}\right)^{\beta/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \text{ and } k(w,r,x) = \left(\frac{\beta w}{\alpha r}\right)^{\alpha/(\alpha+\beta)}\left(\frac{x}{A}\right)^{1/(\alpha+\beta)} \tag{12.38}$$

(c) *Given some initial prices (w^A, r^A, p^A), verify that all cost minimizing bundles lie on the same ray from the origin in the isoquant graph.*

Answer: Dividing the conditional input demands by one another, we get

$$\frac{k(w^A,r^A,x)}{\ell(w^A,r^A,x)} = \frac{\beta w^A}{\alpha r^A}. \tag{12.39}$$

Thus, regardless of what isoquant x we try to reach, the ratio of capital to labor that minimizes the cost of reaching that isoquant is independent of x — implying that all cost minimizing input bundles lie on a ray from the origin.

(d) *If w increases, what happens to the ray on which all cost minimizing bundles lie?*

Answer: If w increases to w', the ratio of capital to labor becomes

$$\frac{\beta w'}{\alpha r^A} > \frac{\beta w^A}{\alpha r^A}; \tag{12.40}$$

i.e. the ray becomes steeper as firms substitute away from labor and toward capital.

(e) *What happens to the profit maximizing input bundles?*

Answer: We see from the input demand equations in (12.35) that both labor and capital demand fall as w increases. (Similarly, we see from equation (12.36) that output supply falls.)

(f) *How do your answers change if w instead decreases?*

Answer: When wage falls to w'', we get that the ray on which cost minimizing bundles occur is

$$\frac{\beta w''}{\alpha r^A} < \frac{\beta w^A}{\alpha r^A}; \tag{12.41}$$

i.e. the ray becomes shallower. From the input demand functions, we also see that demand for labor and capital increase — as does output (as seen in the output supply function).

(g) *If instead p increases, does the ray along which all cost minimizing bundles lie change?*

Answer: The ray along which cost minimizing bundles lie is defined by the ratio of conditional capital to conditional labor demand — which is

$$\frac{k(w,r,x)}{\ell(w,r,x)} = \frac{\beta w}{\alpha r}. \tag{12.42}$$

Since this does not depend on p, we can see that the ray does not depend on output price. This should make sense: Cost minimization does not take output price into account since all it asks is: "what is the least cost way of producing x?"

(h) *Where on that ray will the profit maximizing production plan lie?*

Answer: Since the ray of cost minimizing input bundles remains unchanged, we know that the new profit maximizing plan lies somewhere on that ray. From the output supply equation (12.36), we see that output increases with p. Thus, the new profit maximizing production plan lies above the initial isoquant and on the same ray as the initial profit maximizing production plan.

(i) *What happens to the ray on which all cost minimizing input bundles lie if r falls? What happens to the profit maximizing input bundle?*

Answer: If r falls to r', we get

$$\frac{\beta w^A}{\alpha r'} > \frac{\beta w^A}{\alpha r^A}; \tag{12.43}$$

i.e. the ray on which cost minimizing input bundles lie will be steeper as firms substitute toward capital and away from labor. From the output supply equation (12.36), we can also see that a decrease in r results in an increase in output — thus, the new profit maximizing input bundle lies above the initial isoquant and to the left of the initial ray along which cost minimizing input bundles occurred.

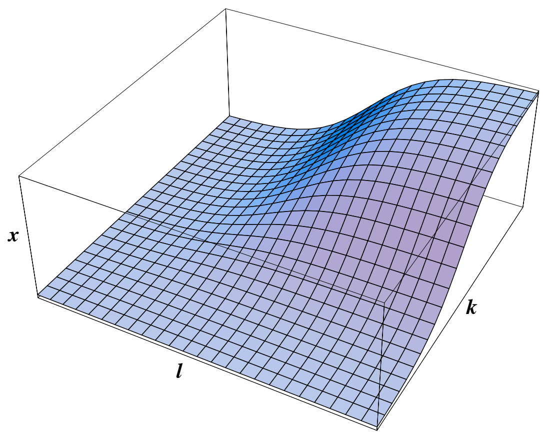
Exercise 12.5

In the absence of recurring fixed costs (such as those in exercise 12.4), the U-shaped cost curves we will often graph in upcoming chapters presume some particular features of the underlying production technology when we have more than 1 input.

A: *Consider the following production technology, where output is on the vertical axis (that ranges from 0 to 100) and the inputs capital and labor are on the two horizontal axes. (The origin on the graph is the left-most corner).*

(a) *Suppose that output and input prices result in some optimal production plan A (that is not a corner solution). Describe in words what would be true at A relative to what we described as an isoprofit plane at the beginning of this chapter.*

Answer: The isoprofit plane $\pi = px - w\ell - rk$ would have to be tangent to the production frontier — with no other portion of the isoprofit plane intersecting the frontier. It is like a



Graph 12.10: Production Frontier with two Inputs

sheet of paper tangent to a "mountain" that is initially getting steeper but eventually becomes shallower. This implies that the isoprofit plane that is tangent at A has a positive vertical intercept.

(b) *Can you tell whether this production frontier has increasing, constant or decreasing returns to scale?*

Answer: The production frontier has initially increasing but eventually decreasing returns to scale — i.e. along every horizontal ray from the origin, the slice of the production frontier has the "sigmoid" shape that we used throughout Chapter 11.

(c) *Illustrate what the slice of this graphical profit maximization problem would look like if you held capital fixed at its optimal level k^A.*

Answer: This is illustrated in panel (a) of Graph 12.11. The tangency of the isoprofit plane shows up as a tangency of the line $x = [(\pi^A + rk^A)/p] + (w/p)\ell$, where the bracketed term is the vertical intercept and the (w/p) term is the slope. (This is just derived from solving the expression $\pi^A = px - w\ell - rk^A$ for x.)

(d) *How would the slice holding labor fixed at its optimal level ℓ^A differ?*

Answer: It would look similar except for re-labeling as in panel (b) of the graph.

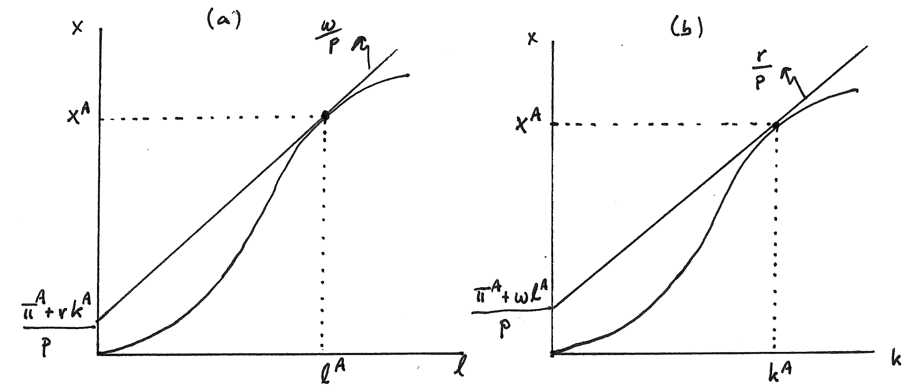
(e) *What two conditions that have to hold at the profit maximizing production plan emerge from these pictures?*

Answer: In panels (a) and (b) of Graph 12.11, the slopes of the isoprofit lines are tangent to the slopes of the production frontier with one of the inputs held fixed. The slope of the production frontier at (ℓ^A, x^A) in panel (a) is the marginal product of labor at that production plan; i.e. MP_ℓ^A. And the slope of the production frontier at (k^A, x^A) in panel (b) is the marginal product of capital at that production plan; i.e. MP_k^A. Thus, the conditions that emerge are

$$MP_\ell^A = \frac{w}{p} \quad \text{and} \quad MP_k^A = \frac{r}{p}. \tag{12.44}$$

(f) *Do you think there is another production plan on this frontier at which these conditions hold?*

Answer: Yes — this would occur on the increasing returns to scale portion of the production frontier where an isoprofit "sheet" is tangent to the lower side of the frontier. This "sheet" will, however, have a negative intercept — implying negative profit.



Graph 12.11: Holding k^A and ℓ^A fixed

(g) *If output price falls, the profit maximizing production plan changes to once again meet the conditions you derived above. Might the price fall so far that no production plan satisfying these conditions is truly profit maximizing?*

Answer: A decrease in p will cause the isoprofit planes to become steeper — causing the profit maximizing production plan to slide down the production frontier as the tangent isoprofit now happens at a steeper slope. This implies that the vertical intercept also slides down — with profit falling. If the price falls too much, this intercept will become negative — implying that the true profit maximizing production plan becomes (0,0,0). Put differently, if the price falls too much, the firm is better off not producing at all rather than producing at the tangency of an isoprofit with the production frontier.

(h) *Can you tell in which direction the optimal production plan changes as output price increases?*

Answer: As output price increases, the isoprofit plane becomes shallower — which implies that the tangency with the production frontier slides up in the direction of the shallower portion of the frontier. Thus, the production plan will involve more of each input and more output.

B: *Suppose your production technology is characterized by the production function*

$$x = f(\ell, k) = \frac{\alpha}{1 + e^{-(\ell-\beta)} + e^{-(k-\gamma)}} \tag{12.45}$$

where e is the base of the natural logarithm. Given what you might have learned in one of the end-of-chapter exercises in Chapter 11 about the function $x = f(\ell) = \alpha/(1 + e^{-(\ell-\beta)})$, can you see how the shape in Graph 12.16 emerges from this extension of this function?

Answer: Even though this question was not meant to be answered directly, the graph given in part A of the question depicts this function for the case where $\alpha = 100$ and $\beta = \gamma = 5$. The graph was generated using the software package Mathematica (as are the other machine generated graphs in some of the answers in this Chapter). As you can see, the function takes on the shape that has initially increasing and eventually diminishing slope along slices holding each input fixed (as well as along rays from the origin.) Note that ℓ and k enter symmetrically given that $\beta = \gamma$ — and the two inputs appear on the axes in the plane from which the surface emanates. The vertical axis in the graph is output x.

(a) *Set up the profit maximization problem.*

Answer: The problem is

$$\max_{x,\ell,k} px - w\ell - rk \text{ subject to } x = \frac{\alpha}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} \tag{12.46}$$

which can also be written as the unconstrained maximization problem

$$\max_{\ell,k} \frac{\alpha p}{1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}} - w\ell - rk. \tag{12.47}$$

(b) *Derive the first order conditions for this optimization problem.*

Answer: We simply take derivatives with respect to w and r and set them to zero. Thus, we get

$$\frac{\alpha p e^{-(\ell-\beta)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = w \text{ and } \frac{\alpha p e^{-(k-\gamma)}}{\left(1+e^{-(\ell-\beta)}+e^{-(k-\gamma)}\right)^2} = r. \tag{12.48}$$

(c) *Substitute $y = e^{-(\ell-\beta)}$ and $z = e^{-(k-\gamma)}$ into the first order conditions. Then, with the first order conditions written with w and r on the right hand sides, divide them by each other and derive from this an expression $y(z,w,r)$ and the inverse expression $z(y,w,r)$.*

Answer: These substitutions lead to the first order conditions becoming

$$\frac{\alpha p y}{(1+y+z)^2} = w \text{ and } \frac{\alpha p z}{(1+y+z)^2} = r. \tag{12.49}$$

Dividing the two equations by each other, we can then derive

$$y(z,w,r) = \frac{wz}{r} \text{ and } z(y,w,r) = \frac{ry}{w}. \tag{12.50}$$

(d) *Substitute $y(z,w,r)$ into the first order condition that contains r. Then manipulate the resulting equation until you have it in the form az^2+bz+c (where the terms a, b and c may be functions of w, r, α and p). (*Hint*: It is helpful to multiply both sides of the equation by r.) The quadratic formula then allows you to derive two "solutions" for z. Choose the one that uses the negative rather than the positive sign in the quadratic formula as your "true" solution $z^*(\alpha,p,w,r)$.*

Answer: Substituting $y(z,w,r)$ into the second expression in equation (12.49) and multiplying both sides by the denominator, we get

$$\alpha p z = r\left(1+\frac{wz}{r}+z\right)^2. \tag{12.51}$$

Multiplying the right hand side by r lets us reduce it to

$$r^2\left(1+\frac{wz}{r}+z\right)^2 = (r+wz+rz)^2 = (r+(w+r)z)^2. \tag{12.52}$$

Thus, when we multiply both sides of equation (12.51) by r, we get

$$\alpha r p z = (r+(w+r)z)^2. \tag{12.53}$$

Expanding the left hand side and grouping terms, we then get

$$(w+r)^2 z^2 + [2r(w+r)-\alpha r p]z + r^2 = 0. \tag{12.54}$$

This is now in the form we need to apply the quadratic formula to solve for z. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$z^*(\alpha,p,w,r) = \frac{-[2r(w+r)-\alpha r p] - \sqrt{[2r(w+r)-\alpha r p]^2 - 4(w+r)^2 r^2}}{2(w+r)^2}. \tag{12.55}$$

(e) *Substitute $z(y,w,r)$ into the first order condition that contains w and then solve for $y^*(\alpha,p,w,r)$ in the same way you solved for $z^*(\alpha,p,w,r)$ in the previous part.*

Answer: Substituting $z(y,w,r)$ into the first expression in equation (12.49) and multiplying both sides by the denominator, we get

$$\alpha py = w\left(1+y+\frac{ry}{w}\right)^2. \tag{12.56}$$

Multiplying the right hand side by w lets us reduce it to

$$w^2\left(1+y+\frac{ry}{w}\right)^2 = (w+wy+ry)^2 = (w+(w+r)y)^2. \tag{12.57}$$

Thus, when we multiply both sides of equation (12.56) by w, we get

$$\alpha wpy = (w+(w+r)y)^2. \tag{12.58}$$

Expanding the left hand side and grouping terms, we then get

$$(w+r)^2y^2 + [2w(w+r)-\alpha wp]z + w^2 = 0. \tag{12.59}$$

This is now in the form we need to apply the quadratic formula to solve for y. The problem tells us to use the version of the formula that has a negative rather than positive sign in front of the square root — thus

$$y^*(\alpha,p,w,r) = \frac{-[2w(w+r)-\alpha wp] - \sqrt{[2w(w+r)-\alpha wp]^2 - 4(w+r)^2w^2}}{2(w+r)^2}. \tag{12.60}$$

(f) *Given the substitutions you did in part (c), you can now write $e^{-(\ell-\beta)} = y^*(\alpha,p,w,r)$ and $e^{-(k-\gamma)} = z^*(\alpha,p,w,r)$. Take natural logs of both sides to solve for labor demand $\ell(w,r,p)$ and capital demand $k(w,r,p)$ (which will be functions of the parameters α, β and γ.)*

Answer: Taking natural logs of $e^{-(\ell-\beta)} = y^*(\alpha,p,w,r)$ and $e^{-(k-\gamma)} = z^*(\alpha,p,w,r)$ gives us

$$-(\ell-\beta) = \ln y^*(\alpha,p,w,r) \text{ and } -(k-\gamma) = \ln z^*(\alpha,p,w,r) \tag{12.61}$$

which can be solved for ℓ and k to get the input demand functions:

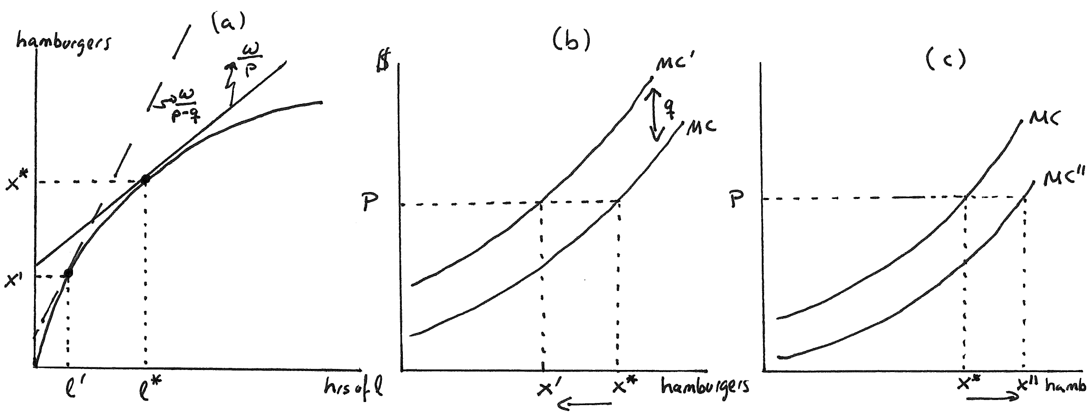
$$\ell(w,r,p) = \beta - \ln y^*(\alpha,p,w,r) \text{ and } k(w,r,p) = \gamma - \ln z^*(\alpha,p,w,r). \tag{12.62}$$

(g) *How much labor and capital will this firm demand if $\alpha = 100$, $\beta = \gamma = 5 = p$, $w = 20 = r$? (It might be easiest to type the solutions you have derived into an Excel spreadsheet in which you can set the parameters of the problem.) How much output will the firm produce? How does your answer change if r falls to $r = 10$? How much profit does the firm make in the two cases.*

Answer: The firm would initially hire approximately 8.035 units of labor and capital to produce 91.23 units of output. When $r = 10$, the optimal production plan would change to $(\ell,k,y) = (8.086, 8.780, 93.59)$ — i.e. the firm would increase production primarily by hiring more capital but also by hiring slightly more labor. Profit is 134.74 in the first case and 218.42 in the second.

(h) *Suppose you had used the other "solutions" in parts (d) and (e) — the ones that emerge from using the quadratic formula in which the square root term is added rather than subtracted. How would your answers to (g) be different — and why did we choose to ignore this "solution"?*

Answer: The solution for the initial values given in part (g) would then have been $(\ell,k,y) \approx (3.35, 3.35, 8.77)$ and this would change to $(\ell,k,y) \approx (2.72, 3.42, 6.41)$ when r falls to 10. This would be an odd outcome — with a drop in the input price r, the problem suggests that output will fall. It is wrong because profit in both cases is negative — meaning these are not profit maximizing production plans. (Profit in the first case is -90.19 and in the second -56.61.)



Graph 12.12: Hamburgers and Grease

Exercise 12.8: *Fast Food Restaurants and Grease*

Everyday and Business Application: *Fast Food Restaurants and Grease: Suppose you run a fast food restaurant that produces only greasy hamburgers using labor that you hire at wage w. There is, however, no way to produce the hamburgers without also producing lots of grease that has to be hauled away. In fact, the only way for you to produce a hamburger is to also produce 1 ounce of grease. You therefore also have to hire a service that comes around and picks up the grease at a cost of q per ounce.*

A: *Since we are assuming that each hamburger comes with 1 ounce of grease that has to be picked up, we can think of this as a single input production process (using only labor) that produces 2 outputs — hamburgers and grease — in equal quantities.*

(a) *On a graph with hours of labor on the horizontal axis and hamburgers on the vertical, illustrate your production frontier assuming decreasing returns to scale. Then illustrate the profit maximizing plan assuming for now that it does not cost anything to have grease picked up (i.e. assume $q = 0$.)*

Answer: This is illustrated in panel (a) of Graph 12.12 where the initial (solid) isoprofit is tangent to the production frontier at production plan (ℓ^*, x^*).

(b) *Now suppose $q > 0$. Can you think of a way of incorporating this into your graph and demonstrating how an increase in q changes the profit maximizing production plan?*

Answer: This is also illustrated in panel (a) of Graph 12.12. Now, the slope of the isoprofit is $w/(p-q)$ because the price per hamburger net of the cost of hauling the associated grease is $(p-q)$ rather than p. As a result, the new isoprofit is tangent at production plan (ℓ', x') — with less output and less labor input than before.

(c) *Illustrate the marginal cost curves with and without q — and then illustrate again how the cost of having grease picked up (i.e. $q > 0$) alters the profit maximizing production choice.*

Answer: The marginal cost curve is upward sloping because of the decreasing returns to scale of the production process. This is illustrated in panel (b) of Graph 12.12 as MC when $q = 0$ and as MC' when $q > 0$. Notice that the MC curve shifts up in a parallel way — because each hamburger now costs q more to produce than before. At a hamburger price of p, this implies that the profit maximizing quantity of hamburgers falls from x^* to x'.

(d) *With increasing fuel prices, the demand for hybrid cars that run partially on gasoline and partially on used cooking grease has increased. As a result, fast food chains report that they no longer have to pay to have grease picked up — in fact, they are increasingly being paid for their grease. (In essence, one of the goods you produce used to have a negative price but* *now has a positive price.) How does this change how many hamburgers are being produced at your fast food restaurant?*

Answer: This is illustrated in panel (c) of Graph 12.12 where the marginal cost curve MC'' is now below the original MC because q is now a "negative" cost. As a result, output increases from x^* to x''.

(e) *We have done all our analysis under the assumption that labor is the only input into hamburger production. Now suppose that labor and capital were both needed in a homothetic, decreasing returns to scale production process. Would any of your conclusions change?*

Answer: Given some input prices w and r, the homotheticity of the production process implies that we will operate on a vertical slice along a ray that emanates from the origin. This slice will look exactly like the production frontier graphed for the single input case in panel (a) of Graph 12.12 and a similar change in the slice of the isoprofits will result in the same conclusion. Similarly, the marginal cost curve will again shift by exactly q — leading to the same pictures as in panels (b) and (c).

(f) *We have also assumed throughout that producing one hamburger necessarily entails producing exactly one ounce of grease. Suppose instead that more or less grease per hamburger could be achieved through the purchase of fattier or less fatty hamburger meat. Would you predict that the increased demand for cooking grease in hybrid vehicles will cause hamburgers at fast food places to increase in cholesterol as higher gasoline prices increase the use of hybrid cars?*

Answer: Yes, as grease turns from being a cost to the firm to being a product that raises revenues, the firm will substitute toward fattier beef — thus increasing the amount of cholesterol in hamburgers.

B: *Suppose that the production function for producing hamburgers x is $x = f(\ell) = A\ell^{\alpha}$ where $\alpha < 1$. Suppose further that, for each hamburger that is produced, 1 ounce of grease is also produced.*

(a) *Set up the profit maximization problem assuming that hamburgers sell for price p and grease costs q (per ounce) to be hauled away.*

Answer: The profit maximization problem is

$$\max_{\ell,x} px - w\ell - qx \text{ subject to } x = A\ell^{\alpha} \tag{12.63}$$

which can also be written as

$$\max_{\ell} (p-q)A\ell^{\alpha} - w\ell. \tag{12.64}$$

(b) *Derive the number of hours of labor you will hire as well as the number of hamburgers you will produce.*

Answer: Differentiating the objective function in equation (12.64) with respect to ℓ and solving for ℓ, we get

$$\ell = \left(\frac{\alpha(p-q)A}{w}\right)^{1/(1-\alpha)}. \tag{12.65}$$

Substituting this into the production function, we get output level

$$x = A\left[\left(\frac{\alpha(p-q)A}{w}\right)^{1/(1-\alpha)}\right]^{\alpha} = A^{1/(1-\alpha)}\left(\frac{\alpha(p-q)}{w}\right)^{\alpha/(1-\alpha)}. \tag{12.66}$$

(c) *Determine the cost function (as a function of w, q and x).*

Answer: Inverting the production function, we get the conditional labor demand function

$$\ell(w,x) = \left(\frac{x}{A}\right)^{1/\alpha}. \tag{12.67}$$

The cost function is then simply the conditional labor demand function multiplied by the cost of labor w plus the cost of hauling away the grease; i.e.

$$C(w,x,q) = w\left(\frac{x}{A}\right)^{1/\alpha} + qx. \tag{12.68}$$

(d) *Derive from this the marginal cost function.*

Answer: Taking the derivative of the cost function with respect to x, we get

$$MC(w,x,q) = \left(\frac{w}{\alpha A^{1/\alpha}}\right) x^{(1-\alpha)/\alpha} + q. \tag{12.69}$$

(e) *Use the marginal cost function to determine the profit maximizing number of hamburgers and compare your answer to what you got in (b).*

Answer: Setting the marginal cost function equal to price p and solving for x, we get

$$x = A^{1/(1-\alpha)} \left(\frac{\alpha(p-q)}{w}\right)^{\alpha/(1-\alpha)} \tag{12.70}$$

which is identical to what we derived in (b).

(f) *How many hours of labor will you hire?*

Answer: Plugging our result for x back into the conditional labor demand function in equation (12.67), we get

$$\ell = \left(\frac{\alpha(p-q)A}{w}\right)^{1/(1-\alpha)} \tag{12.71}$$

which is again identical to what we derived in part (b).

(g) *How does your production of hamburgers change as grease becomes a commodity that people will pay for (rather than one you have to pay to have hauled away)?*

Answer: It is easy to see from our output equation that, as q becomes smaller and turns negative, output increases.

Conclusion: Potentially Helpful Reminders

1. Profit maximization implies that marginal product equals input price for *ALL* inputs. Short run profit maximization therefore implies just that $MP_\ell = w$, while long run profit maximization implies that both $MP_\ell = w$ and $MP_k = w$.

2. Cost minimization implies that $TRS = -w/r$ which, since $-TRS = MP_\ell/MP_k$, is equivalent to saying $MP_\ell/MP_k = w/r$. You should be able to show that the profit maximization conditions ($MP_\ell = w$ and $MP_k = w$) imply that the cost minimization condition holds, but the reverse does not hold.

3. Profit maximization can be seen graphically as a tangency of the vertical production frontier slices that hold one input fixed with the slice of the isoprofit plane. You should understand how that tangency is equivalent to saying $MP_\ell = w$ and $MP_k = w$.

4. Cost minimization can be seen graphically as tangencies of isocosts and isoquants. You should understand how the condition $-TRS = MP_\ell/MP_k = w/r$ must logically hold at all production plans that minimize cost. You should also understand that, when the production frontier is homothetic, *ALL* such tangencies will happen along a single ray from the origin for a given w and r. And you should understand why typically only one of those tangecies represents a *profit maximizing* production plan.

5. End-of-Chapter problem 12.1 is a good problem to practice with concepts contained in the above points — and a good problem to use for preparation for Chapter 13.

6. At the end of Chapter 11, we showed that the supply curve is the part of the marginal cost curve that lies above average cost. The same is true in this chapter when there are 2 inputs — and the same will always be true, in the short and long run, so long as we define costs correctly.

7. One of the points emphasized in end-of-chapter exercises (but only partially emphasized in the text chapter) is that U-shaped average cost curves can arise in one of two ways: (1) because of production technologies that initially exhibit increasing returns to scale but eventually turn to decreasing returns to scale; and (2) because of the existence of a recurring *fixed cost*. This idea is further developed in end-of-chapter exercise 12.4 and then in Chapter 13.