CHAPTER

3

Choice Sets in Labor and Financial Markets

This chapter is a straightforward extension of Chapter 2 where we had shown that budget constraints can arise from someone owning an **endowment** that he can sell to generate the income needed for purchasing a different consumption bundle. That's exactly what workers and savers do: Workers own their time and can sell it to earn income, and savers own some investment that they can sell and turn into consumption. (Borrowers, on the other hand, own some future asset — such as the income they can earn in the future — that they can sell.) Once you understand how budgets can arise from stuff we own, it becomes straightforward to think about workers and savers/borrowers.

Chapter Highlights

The main points of the chapter are:

1. **Wages** and **interest rates** are prices in particular markets — and therefore give rise to the opportunity costs we face when making choices in those markets as leisure time or investments are sold.

2. When budgets arise from the sale of endowments, price increases no longer unambiguously shrink budgets nor do price decreases unambiguously increase budgets **as budget lines rotate through the endowment bundle**.

3. **Endowment bundles** are those that can always be consumed regardless of what prices (or wages or interest rates) emerge in the economy.

4. **Government policies** can change the economic incentives faced by workers and savers by changing the choice sets they face.

5. An amount \$X in the future has a **present value** less than \$X — because borrowing on that amount to consume more now entails paying interest.

6. The 2-dimensional models of leisure/consumption and intertemporal budgets are really just **"slices" of higher dimensional budgets** along which some things are held fixed.

Using the LiveGraphs

For an overview of what is contained on the LiveGraphs site for each of the chapters (from Chapter 2 through 29) and how you might utilize this resource, see pages 2-3 of Chapter 1 of this *Study Guide*. To access the LiveGraphs for Chapter 3, click the *Chapter 3* tab on the left side of the LiveGraphs web site.

As I am writing this, we do not yet have any plans for *Exploring Relationships* modules for this Chapter. However, if you are having any difficulty at all with the concept that our 2-dimensional graphs are coming out of higher-dimensional models that hold some aspects of the problem fixed, I urge you to play the animated graphic for Graph 3.5. This graphic shows how, for instance, an intertemporal consumption budget set is contained within a large choice set that also contains a choice of how much to work.

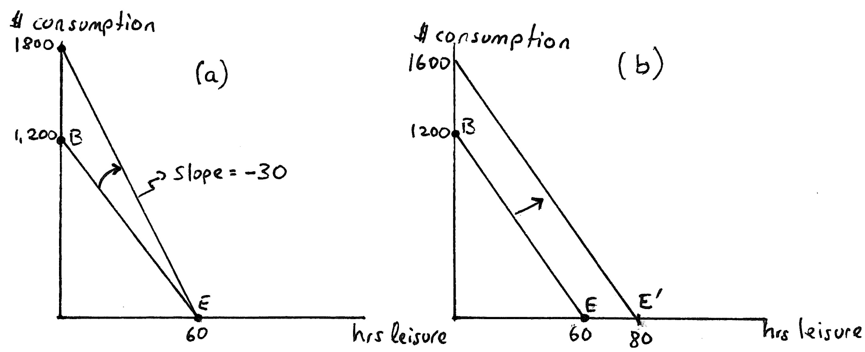
3A Solutions to Within-Chapter Exercises for Part A

Exercise 3A.1 *Illustrate what happens to the original budget constraint if your wage increases to \$30 per hour. What if your friend instead introduces you to caffeine which allows you to sleep less and thus take up to 80 hours of leisure time per week?*

Answer: If the wage goes up to \$30 per hour, you could earn as much as \$1,800 per week if you spent all 60 hours working. Thus, the consumption-intercept goes to 1,800, but the endowment point E has not changed. The resulting budget constraint (graphed in panel (a) of Graph 3.1 on the next page) therefore rotates clockwise through E and has a new slope equal to $-w = -30$. If, on the other hand, the endowment of leisure goes up to 80, the budget shifts parallel as in panel (b) of the graph. The slope remains at $-w = -20$ since the wage has not changed.

Exercise 3A.2 *Verify the dollar quantities on the axes in Graph 3.3a-c.*

Answer: In panel (a), you have \$10,000 now, which places your endowment point on the horizontal axis at \$10,000. At a 10% interest rate, you would have earned \$1,000 in interest if you consumed nothing today and you saved everything. That would leave you with \$11,000 of consumption next year at B. When the interest rate falls to 5%, the most in interest you can earn is \$500, leaving you with



Graph 3.1: (a) An increase in w; (b) An increase in Leisure

\$10,500 next year if you consume nothing this year. This is the relevant intercept at B'.

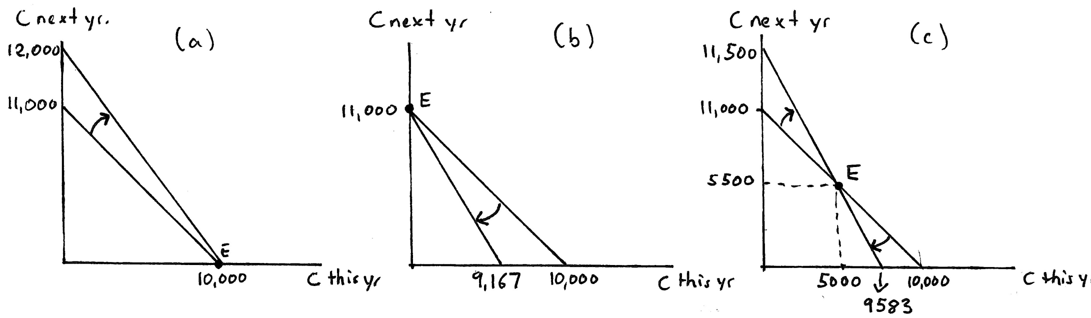
In panel (b), you earn \$11,000 next year but nothing now — thus placing your endowment point on the vertical axis at \$11,000. When you borrow on next year's income in order to consume this year, the most the bank will lend you is an amount that, when paid back with interest, will be equal to what you earn next year. When the interest rate is 10%, the bank would then be willing to lend you $\$11,000/(1+0.1) = \$10,000$. If you ended up borrowing \$10,000, you would then owe the bank \$10,000 plus interest of \$1,000 for a total of \$11,000. Thus, at a 10% interest rate, the most you can consume this year is \$10,000 if you are willing to not consume at all next year (bundle A). When the interest rate falls to 5%, the bank would be willing to lend you up to \$11,000/(1+1.05)=\$10,476.19 (bundle A').

In panel (c), you earn \$5,000 now and \$5,500 next year — making that your endowment point. Were you to save all your current income at a 10% interest rate, you could have \$5,500 in the bank next year — which, together with your \$5,500 income next year, would allow you total consumption of \$11,000 (bundle B). If, on the other hand, you decide to do all your consumption this year, you can borrow \$5,500/(1+0.1)=\$5,000 — which, together with this year's income of \$5,000, leaves you with \$10,000 in consumption this year (bundle A). At a 5% interest rate, on the other hand, you would accumulate \$5,000(1+0.05)=\$5,250 in your savings account by saving all your current income, leaving you with a total of \$10,750 (bundle B') next year when next year's income of \$5,500 is added. Or you can consume all now, with the bank lending you \$5,500/(1+0.05)=\$5,238.10 that, together with this year's income of \$5,000, lets you consume \$10,238 (bundle A') now.

Exercise 3A.3 *In each of the panels of Graph 3.3, how would the choice set change if the interest rate went to 20%?*

Answer: Since the interest rate is higher, the slopes would all become steeper

— with the constraints rotating through the relevant endowment bundle. These changes are depicted in panels (a) through (c) of Graph 3.2. For the reasoning behind the intercepts, see the previous within-chapter-exercise.



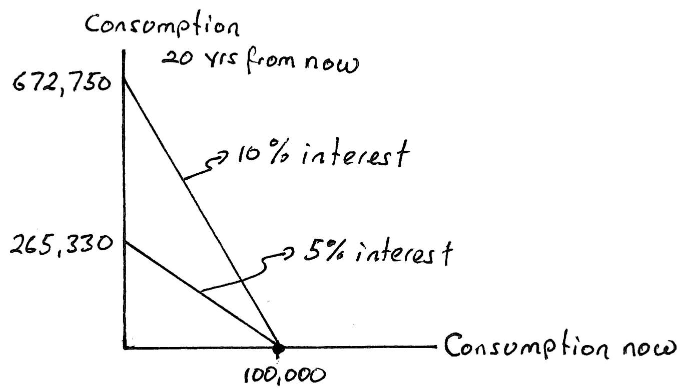
Graph 3.2: An increase in the interest rate to 20%

Exercise 3A.4 *So far, we have implicitly assumed that interest compounds yearly —- i.e. you begin to earn interest on interest only at the end of each year. Often, interest compounds more frequently. Suppose that you put $10,000 in the bank now at an annual interest rate of 10% but that interest compounds monthly rather than yearly. Your monthly interest rate is then 10/12 or 0.833%. Defining n as the number of months and using the information in the previous paragraph, how much would you have in the bank after 1 year? Compare this to the amount we calculated you would have when interest compounds annually.*

Answer: You would have $10000(1+r)^n = 10000(1.00833)^{12} = 11,047.13$. Thus, over a 1 year period, by putting $10,000 into a savings account at 10% annual interest that compounds monthly, you get $47.13 more in interest than when putting the same amount into a savings account at the same interest rate but compounded annually.

Exercise 3A.5 *Suppose you just inherited $100,000 and you are trying to choose how much of this to consume now and how much of it to save for retirement 20 years from now. Illustrate your choice set with "dollars of consumption now" and "dollars of consumption 20 years from now" assuming an interest rate of 5% (compounded annually). What happens if the interest rate suddenly jumps to 10% (compounded annually)?*

Answer: Regardless of the interest rate, you can choose to consume the entire $100,000 now (which therefore is your endowment point that lies on the horizontal axis). If you save all of it, you will collect $100000(1+r)^{20}$ where r is 0.05 when the interest rate is 5% and 0.1 when the interest rate is 10%. This results in intercepts on the vertical axis of $265,330 if the interest rate is 5% and $672,750 if the interest rate is 10%. This is depicted in Graph 3.3 (on the next page).



Graph 3.3: Consume now or 20 years from now

Exercise 3A.6 *Draw a budget constraint similar to Graph 3.5 assuming you do not work this summer but rather next summer at a wage of $22 per hour (with a total possible number of leisure hours of 600 next summer) and assuming that the interest rate is 5%. Where is the 5% interest rate budget line from Graph 3.3b in the graph you have just drawn?*

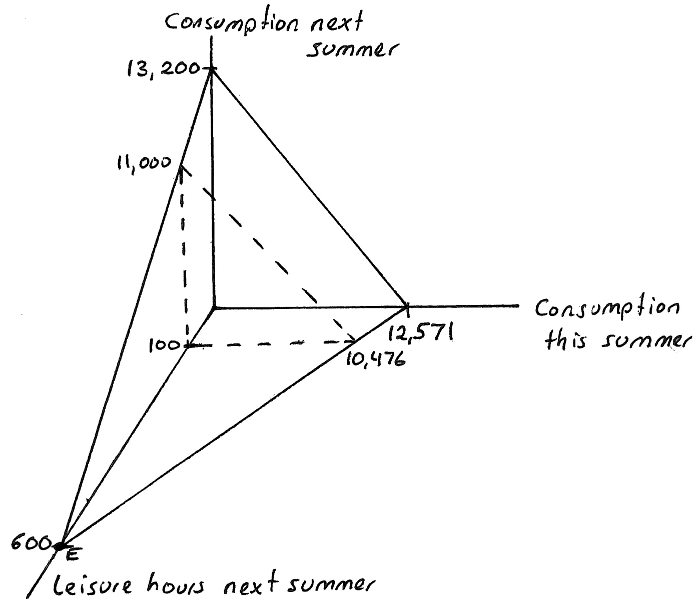
Answer: The graph is depicted in Graph 3.4 (on the next page), with the dashed slice equivalent to the budget line in the 2-dimensional graph earlier in the text. The endowment point is once again leisure — only this time 600 leisure hours *next* summer. At $22 per hour, this translates into $13,200 of consumption next summer if you choose to work all 600 hours. If you were to work all 600 hours but you wanted to borrow and consume all the resulting income now, your could consume $\$13,200/(1+0.05) = \$12,571$. The 2-dimensional graph earlier on took $11,000 of income next summer as the starting point, thus implicitly assuming that you have chosen to work for 500 hours next summer, leaving you with 100 hours of leisure. Thus, the slice at 100 hours of leisure next summer represents the budget line in Graph 3.3b in the textbook.

3B Solutions to Within-Chapter Exercises for Part B

Exercise 3B.1 *Graph the choice set in equation (3.5) when n=2,* $p_1 = 1$, $p_2 = 2$, $w = 20$ *and* $L = 60$.

Answer: Using the values given in the exercise, the choice set is defined as

$$\left\{(x_1, x_2, \ell) \in \mathbb{R}^3_+ \mid x_1 + 2x_2 \leq 20(60 - \ell)\right\}. \tag{3.1}$$



Graph 3.4: Consuming over 2 summers but working only next summer

This is depicted in Graph 3.5 (on the next page). The endowment point is leisure of 60 hours with no consumption. If this worker works all the time, she will earn \$1,200 given that she earns a wage of \$20 per hour. With that, she could buy as many as 1,200 units of x_1 if that was all she bought (as $p_1 = 1$), or as many as 600 units of x_2 if that is all she bought (at $p_2 = 2$).

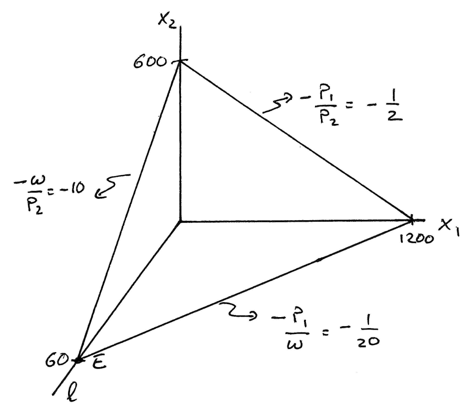
Exercise 3B.2 *Translate the choice sets graphed in Graph 3.2 into mathematical notation defining the choice sets.*

Answer: The choice set in panel (a) of the textbook Graph 3.2 is

$$\{\,(\ell,c)\in\mathbb{R}^2_+ \mid \quad c = 1000-10\ell \quad \text{for } \ell \le 20 \text{ and } \quad c = 1200-20\ell \quad \text{for } \ell > 20\,\}, \tag{3.2}$$

while the choice set in panel (b) is

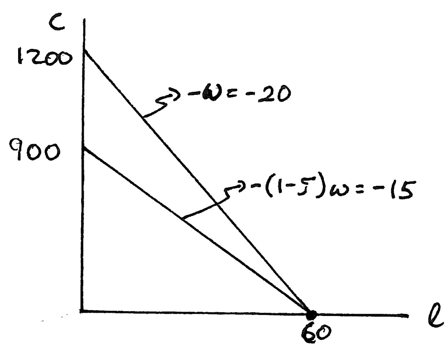
$$\{\,(\ell,c)\in\mathbb{R}^2_+ \mid \quad c = 1400-30\ell \quad \text{for } \ell \le 20 \text{ and } \quad c = 1200-20\ell \quad \text{for } \ell > 20\,\}. \tag{3.3}$$



Graph 3.5: Leisure and 2 goods

Exercise 3B.3 *Suppose* $w = 20$ *and* $L = 60$. *Graph the budget constraint in the absence of taxes. Then suppose a wage tax* $t = 0.25$ *is introduced. Illustrate how this changes your equation and the graph.*

Answer: The budget equation is given by $c = (1-t)w(L-\ell)$ which is $c = 20(60-\ell) = 1200 - 20\ell$ when $t = 0$ and $c = (1-0.25)20(60-\ell) = 900 - 15\ell$ when $t = 0.25$. These are depicted in Graph 3.6 below.



Graph 3.6: 25% tax on wages

Exercise 3B.4 *How would the budget line equation change if, instead of a tax on wages, the government imposed a tax on all consumption goods such that the tax paid by consumers* *equaled 25% of consumption. Show how this changes the equation and the corresponding graph of the budget line.*

Answer: Since all income earned through wages is by definition consumed in this model, a tax equivalent to 25% of consumption is equivalent to a 25% wage tax. So nothing would change in the equation or graph.

Exercise 3B.5 *Suppose* $(e_1 - c_1)$ *is negative — i.e. suppose you are borrowing rather than saving in period 1. Can you still make intuitive sense of the equation?*

Answer: When $(e_1 - c_1)$ is negative, you have consumed your period 1 endowment e_1 plus an amount $(c_1 - e_1)$ on top of it. The only way you could consume more than you had in period 1 is to borrow from period 2 — thus you must have borrowed the amount $(c_1 - e_1)$. One year later, you have to pay back that amount plus interest for a total of $(1+r)(c_1 - e_1)$. We therefore have to subtract that amount from e_2 to determine how much you will have left after paying back what you owe to the bank. Thus, consumption c_2 in period 2 must be no more than $e_2 - (1+r)(c_1 - e_1)$, which can be re-written as $c_2 \leq (1+r)(e_1 - c_1) + e_2$. In the case of borrowing, the quantity $(1+r)(e_1 - c_1)$ is therefore negative and equal to what you owe the bank in period 2.

Exercise 3B.6 *Use the information behind each of the scenarios graphed in Graph 3.3 to plug into equation (3.8) that scenario's relevant values for* e_1*,* e_2 *and* r*. Then demonstrate that the budget lines graphed are consistent with the underlying mathematics of equation (3.8), and more generally, make intuitive sense of the intercept and slope terms as they appear in equation (3.8).*

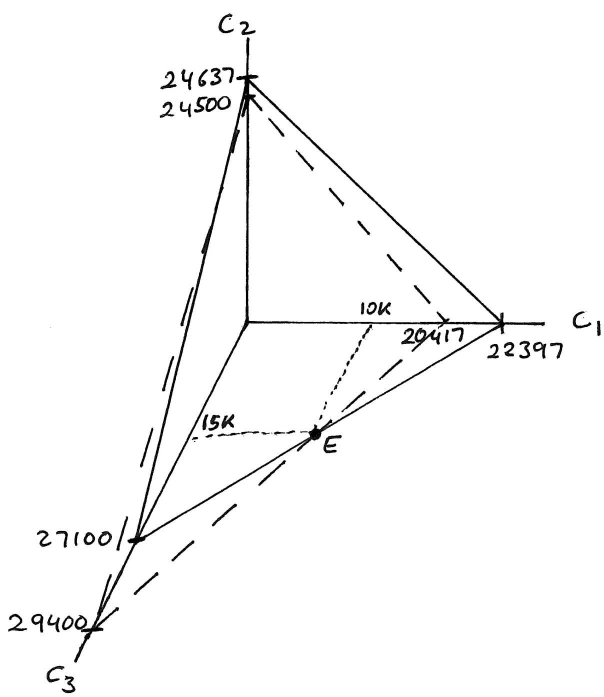
Answer: In panel (a) of the textbook Graph 3.3, $(e_1, e_2) = (10000, 0)$ and the interest rate is initially $r = 0.1$ and then falls to $r = 0.05$. Plugging these into the equation, we get $c_2 \leq 10000(1+r) - (1+r)c_1$. The intercept term is then 11,000 when $r = 0.1$ and 10,500 when $r = 0.05$, and the slopes are analogously either 1.10 or 1.05. This is precisely what is graphed in the textbook.

In panel (b) of the textbook Graph 3.3, $(e_1, e_2) = (0, 11000)$. The equation then becomes $c_2 \leq 11000 - (1+r)c_1$. Thus, regardless of the interest rate, the intercept is 11,000, but the slope is 1.10 when $r = 0.1$ and 1.05 when $r = 0.05$, all as depicted in the graphs in the text.

Finally, in panel (c) of the textbook Graph 3.3, $(e_1, e_2) = (5000, 5500)$. When plugged into the equation, this becomes $c_2 \leq (1+r)5000 + 5500 - (1+r)c_1$. This gives an intercept of $1.1(5000) + 5500 = 11000$ when $r = 0.10$ and $1.05(5000) + 5500 = 10750$ when $r = 0.05$, with slopes of 1.10 and 1.05 — all as depicted in the graph in the text.

For an intuitive explanation of all this, see the answer to within-chapter exercise 3A.2.

Exercise 3B.7 *Suppose you expect to earn* \$10,000 *this summer,* \$0 *next summer and* \$15,000 *two summers from now. Using* c_1*,* c_2*, and* c_3 *to denote consumption over these three summers, write down your budget constraint assuming an annual (and annually compounding) interest* *rate of 10%. Then illustrate this constraint on a three dimensional graph with c_1, c_2, and c_3 on the three axes. How does your equation and graph change if the interest rate increases to 20%?*



Graph 3.7: Income now and 2 years from now

Answer: Graph 3.7 depicts the constraint when the interest rate is 10% in solid lines and the constraint when the interest rate is 20% in dashed lines. Both constraints pass through the endowment point E. The underlying equation for each budget plane is

$$c_3 + (1+r)c_2 + (1+r)^2 c_1 = 15000 + 10000(1+r)^2. \tag{3.4}$$

Consider first how this relates to the solid budget constraint when the interest rate is $r = 0.10$. To determine the c_3 intercept, we set $c_1 = c_2 = 0$ and get that $c_3 = 15000 + 10000(1+0.1)^2) = 27100$. To determine the c_2 intercept, we set $c_1 = c_3 = 0$ and get $(1+0.1)c_2 = 27100$ or $c_2 = 27100/1.1 = 24637$. Finally, to get the c_1 intercept, we set $c_3 = c_2 = 0$ and get $(1+0.1)^2 c_1 = 27100$ or $c_3 = 22397$. Notice that, focusing simply on the slice of the graph that holds $c_3 = 0$, we see that the c_2 intercept (24,637) divided by the c_1 intercept (22,397) is 1.1 or $(1+r)$ — which is exactly the slope we would expect for a one-period intertemporal budget constraint. The same is true for the slope on the slice that holds $c_1 = 0$. And, for the bottom slice where $c_2 = 0$, the slope is 1.21 or $(1+r)^2$ — again what we would expect for a 2-period intertemporal budget constraint.

The intercepts on the dashed budget plane can be similarly calculated, this time substituting $r = 0.2$ rather than $r = 0.1$ into the equations.

Exercise 3B.8 *When $L = 600$, $w = 20$ and $r = 0.1$, show how the equation above translates directly into Graph 3.5.*

Answer: Plugging in these values, we get $1.1c_1 + c_2 = 1.1(20)(600 - \ell)$ which can also be written as

$$1.1c_1 + c_2 + 22\ell = 13200. \tag{3.5}$$

To determine the intercept on the c_1 axis, we simply set $c_2 = \ell = 0$ and get $1.1c_1 = 13200$ or $c_1 = 12000$. To determine the intercept on the c_2 axis, we set $c_1 = \ell = 0$ and get $c_2 = 13200$. Finally, we can check that the ℓ intercept is equal to the leisure endowment — by plugging $c_1 = c_2 = 0$ in to get $22\ell = 13200$ or $\ell = 600$.

Exercise 3B.9 *Define mathematically a generalized version of the choice set in expression (3.18) under the assumption that you have both a leisure endowment L_1 this summer and another leisure endowment L_2 next summer. What is the value of L_2 in order for Graph 3.5 to be the correct 3-dimensional "slice" of this 4-dimensional choice set?*

Answer: The budget set would then be defined as

$$B(L_1, L_2, w, r) = \{(c_1, c_2, \ell_1, \ell_2) \in \mathbb{R}^4_+ \mid (1+r)c_1 + c_2 = (1+r)w(L_1 - \ell_1) + w(L_2 - \ell_2)\}. \tag{3.6}$$

When $L_2 = \ell_2$, the equation inside the set becomes

$$(1+r)c_1 + c_2 = (1+r)w(L_1 - \ell_1), \tag{3.7}$$

which is exactly the 3-dimensional choice set referred to in the text (where it was implicitly assumed that you consume all your leisure next summer and thus derive no income next summer).

End of Chapter Exercises

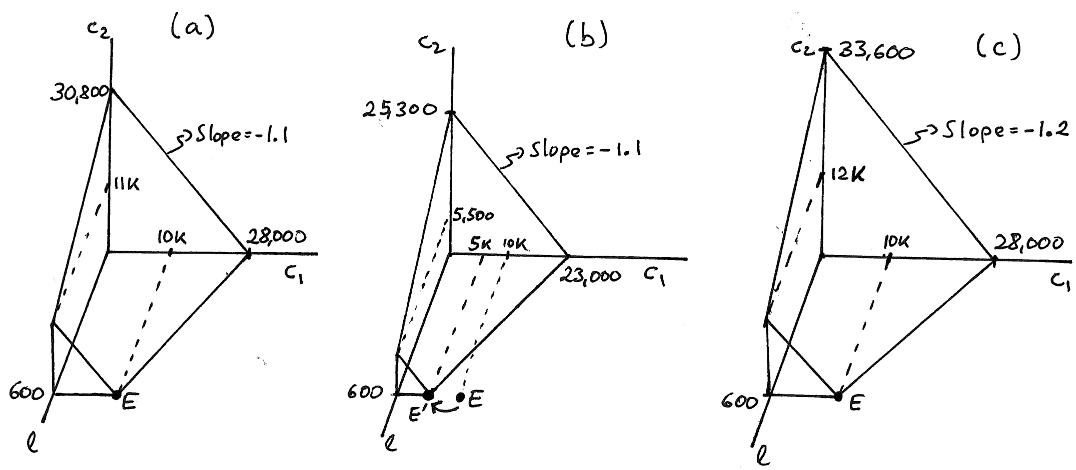
Exercise 3.3

You have \$10,000 sitting in a savings account, 600 hours of leisure time this summer and an opportunity to work at a \$30 hourly wage.

A: *Next summer is the last summer before you start working for a living, and so you plan to take the whole summer off and relax. You need to decide how much to work this summer and how much to spend on consumption this summer and next summer. Any investments you make for the year will yield a 10% rate of return over the coming year.*

(a) *On a three dimensional graph with this summer's leisure (ℓ), this summer's consumption (c_1) and next summer's consumption (c_2) on the axes, illustrate your endowment point as well as your budget constraint. Carefully label your graph and indicate where the endowment point is.*

Answer: This is graphed in panel (a) of Graph 3.8. The endowment point — the point on the budget constraint that is always available regardless of prices — is $(\ell, c_1, c_2) = (600, 10000, 0)$ where no work is done and all $10,000 in the savings account is consumed now. No matter what the wage or the interest rate is, you can always consume this bundle. However, you can also work up to 600 hours this summer, which would earn you up to an additional $18,000 for consumption now. So the most you could consume this summer if you worked all the time and emptied your savings account is $28,000, the c_1 intercept. You can't consume any more than 600 hours of leisure — so 600 is the ℓ intercept. If you earned $18,000 this summer (by working all the time and thus consuming zero leisure), and if you consumed nothing this summer, then the most you could consume next summer is $28,000 times $(1+r)$ where $r = 0.1$ is the interest rate. This gives the c_2 intercept of $30,800. If you don't work (i.e. $\ell = 600$) and you consume nothing this summer (i.e. $c_1 = 0$), then you simply have your $10,000 from your savings account plus $1,000 in interest next summer, for a total of $11,000 in c_2. The overall shape of the budget constraint then becomes the usual triangular shape but, because you can't buy leisure beyond 600 hours with your savings, the tip of the triangle is cut off.



Graph 3.8: Leisure/Consumption and Intertemporal Tradeoffs Combined

(b) *How does your answer change if you suddenly realize you still need to pay $5,000 in tuition for next year, payable immediately?*

Answer: This is graphed in panel (b). Since $5,000 has to leave your savings account right now, this leaves you with only $5,000 in the account and thus shifts your endowment point in. The rest of the budget constraint is derived through similar logic to what was used above. The budget constraint is again a triangular plane with the tip cut off, only now the tip that's cut off is smaller. (Had the immediately-due tuition payment been $10,000, the cut-off tip would have disappeared.)

(c) *How does your answer change if instead the interest rate doubles to 20%?*

Answer: This is illustrated in panel (c) of Graph 3.8 where E is back to what it was in part (a) but the amount of consumption next summer goes up because of the higher interest rate. The reasoning for the various intercepts is similar to that above.

(d) *In (b) and (c), which slopes are different than in (a)?*

Answer: Slopes are formed by ratios of prices — so the only way that slopes can change is if a price has changed. In the scenario in (b), no price has changed. Thus, the only thing that happens is that E shifts in as indicated in the graph. This changes the various intercepts, but the slopes in each plane are parallel to those from panel (a). In the scenario in (c), on the other hand, the interest rate changes. The interest rate is a price that is reflected in any intertemporal budget constraint — i.e. any budget constraint that spans across time periods. In our graph in panel (c), this includes the constraint that lies in the plane that shows the tradeoff between c_1 and c_2, and the plane that shows the tradeoff between ℓ (which is leisure *this* summer) and c_2. Those slopes change as the interest rate changes, but the slope in the plane that illustrates the tradeoff between ℓ and c_1 does not — because that tradeoff happens within the same time period and thus does not involve interest payments.

B: *Derive the mathematical expression for your budget constraint in 3.3A and explain how elements of this expression relate to the slopes and intercepts you graphed.*

Answer: An intuitive way to construct this mathematical expression involves thinking about how much consumption c_2 is possible next summer. If you consume ℓ amount of leisure this summer, you will have a total of $10000+30(600-\ell)$ available for consumption *this* summer — the \$10,000 in the savings account plus your earnings (at a wage of \$30) from hours that you did not consume as leisure. When we take this amount and subtract from it the consumption c_1 you actually undertake this summer, we get the amount that will be in the savings account for the year to accumulate interest. Thus, next summer, you will have $10000+30(600-\ell)-c_1$ times $(1+r)$ (where $r=0.1$ is the interest rate); i.e.

$$c_2 = 1.1\,[10000+(30)(600-\ell)-c_1\,] = 30800-33\ell-1.1c_1\,, \tag{3.8}$$

or, written differently,

$$1.1c_1 + c_2 + 33\ell = 30800. \tag{3.9}$$

The only caveat to this when we define the budget plane is that we have to take into account that you cannot consume more than 600 hours of leisure (or negative quantities of consumption). We can incorporate this by defining the budget plane as

$$\left\{(c_1, c_2, \ell) \in \mathbb{R}^3_+ \mid 1.1c_1 + c_2 + 33\ell = 30800 \text{ and } \ell \le 600\right\}. \tag{3.10}$$

Exercise 3.11

Business Application: *Compound Interest over the Long Run: Uncle Vern has just come into some money — \$100,000 — and is thinking about putting this away into some investment accounts for a while.*

A: *Vern is a simple guy — so he goes to the bank and asks them what the easiest option for him is. They tell him he could put it into a savings account with a 10% interest rate (compounded annually).*

(a) *Vern quickly does some math to see how much money he'll have 1 year from now, 5 years from now, 10 years from now and 25 years from now assuming he never makes withdrawals. He doesn't know much about compounding — so he just guesses that if he leaves the money in for 1 year, he'll have 10% more; if he leaves it in 5 years at 10% per year he'll have 50% more; if he leaves it in for 10 years he'll have 100% more and if he leaves it in for 25 years he'll have 250% more. How much does he expect to have at these different times in the future?*

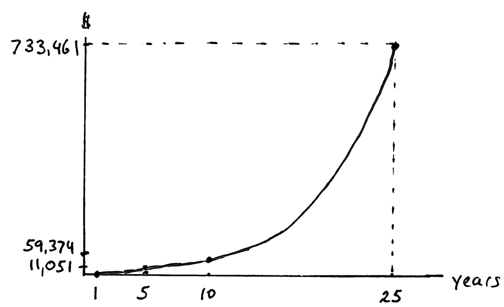
Answer: He expects to have \$110,000 1 year from now, \$150,000 five years from now, \$200,000 ten years from now and \$350,000 twenty-five years from now.

(b) *Taking the compounding of interest into account, how much will he really have?*

Answer: Using our usual formula, the actual balance n years from now is \$100000$(1.1)^n$. This gives \$110,000 one year from now, \$161,051 five years from now, \$259,374.25 ten years from now, and \$1,083,460.59 twenty five years from now.

(c) *On a graph with years on the horizontal axis and dollars on the vertical, illustrate the size of Vern's error for the different time intervals for which he calculated the size of his savings account.*

Answer: The size of the error is \$0 one year from now, \$11,051 five years from now, \$59,374.25 ten years from now and \$733,460.59 twenty-five years from now. This is graphed in Graph 3.9.



Graph 3.9: Error from not compounding over time

(d) *True/False: Errors made by not taking the compounding of interest into account expand at an increasing rate over time.*

Answer: The statement is clearly true based on the answers above.

B: *Suppose that the annual interest rate is r.*

(a) *Assuming you will put x into an account now and leave it in for n years, derive the implicit formula Vern used when he did not take into account interest compounding.*

Answer: Letting y_n denote the amount he projected will be in the savings account n years from now, he used the formula $y_n = x(1+nr)$.

(b) *What is the correct formula that includes compounding.*

Answer: Using z_n to determine the actual amount in the savings account n years from now, the correct formula is $z_n = x(1+r)^n$

(c) *Define a new function that is the difference between these. Then take the first and second derivatives with respect to n and interpret them.*

Answer: The new function is $z_n - y_n = x(1+r)^n - x(1+nr)$. First, note that, when $n=1$, this function reduces to zero — implying the difference between Vern's prediction and reality is zero 1 year from now (just as you determined earlier in the problem). The derivative of this function with respect to n is

$$\frac{\partial(z_n - y_n)}{\partial n} = nx(1+r)^{n-1} - xr = x\left[n(1+r)^{n-1} - r\right]. \tag{3.11}$$

For any $n \geq 1$, this is clearly positive — which means the difference is increasing with time. The second derivative with respect to n is $x(n-1)n(1+r)^{n-2}$ which is positive for $n > 1$. Thus the rate at which the difference increases is increasing with time. This is exactly what we illustrated in Graph 3.9.

Exercise 3.14

Business Application: *A Trick for Calculating the Value of Annuities*: *In several of the above exercises, we have indicated that an infinite series* $1/(1+r)+1/(1+r)^2+1/(1+r)^3+...$ *sums to* $1/r$. *This can (and has, in some of the B-parts of exercises above) been used to calculate the value of an annuity that pays* x *per year starting next year and continuing every year eternally as* x/r.

A: *Knowing the information above, we can use a trick to calculate the value of annuities that do not go on forever. For this example, consider an annuity that pays $10,000 per year for* 10 *years beginning next year, and assume* $r = 0.1$.

(a) *First, calculate the value of an annuity that begins paying $10,000 next year and then every year thereafter (without end).*

Answer: The value of such an annuity would be

$$\frac{\$10,000}{1.1}+\frac{\$10,000}{1.1^2}+\frac{\$10,000}{1.1^3}+...=\frac{\$10,000}{0.1}=\$100,000. \tag{3.12}$$

(b) *Next, suppose you are given such an annuity in 10 years; i.e. suppose you know that the first payment will come 11 years from now. What is the consumption value of such an annuity today?*

Answer: We know that such an annuity is worth $100,000 the year before the payments start; i.e. in 10 years. Anything that is worth $100,000 ten years from now is worth only $\$100,000/(1.1)^{10} = \$38,554.33$ today.

(c) *Now consider this: Think of the 10-year annuity as the difference between an infinitely lasting annuity that starts making payments next year and an infinitely lasting annuity that starts 11 years from now. What is the 10-year annuity worth when you think of it in these terms?*

Answer: The infinitely-lasting annuity with payments starting next year is worth $100,000. The infinitely-lasting annuity with payments starting 11 years from now is worth $38,554.33. Thus, a 10-year annuity with payments starting next year is worth $\$100,000 - \$38,554.33 = \$61,445.67$.

(d) *Calculate the value of the same 10-year annuity without using the trick above. Do you get the same answer?*

Answer: Without using the trick, we would have to calculate this as

$$\frac{\$10,000}{1.1}+\frac{\$10,000}{1.1^2}\frac{\$10,000}{1.1^3}+...+\frac{\$10,000}{1.1^{10}}, \tag{3.13}$$

which sums to $61,445.67. It should make sense that these two methods result in the same outcome. If we had not known the sum of the infinite sequence described at the beginning of the problem, we would have calculated the value of an infinitely-lasting annuity with payments starting next year as

$$\left[\frac{\$10,000}{1.1}+\frac{\$10,000}{1.1^2}+\frac{\$10,000}{1.1^3}+...+\frac{\$10,000}{1.1^{10}}\right]+\left[\frac{\$10,000}{1.1^{11}}+\frac{\$10,000}{1.1^{12}}+...\right], \tag{3.14}$$

and the value of an infinitely-lasting annuity with payments starting 11 years from now would have been calculated as

$$\left[\frac{\$10,000}{1.1^{11}}+\frac{\$10,000}{1.1^{12}}+...\right]. \tag{3.15}$$

Subtracting the equation (3.15) from (3.14) then gives equation (3.13). The trick of viewing a finite annuity as the difference between two infinitely lasting annuities therefore gives us precisely what we know the formula for a finitely lasting annuity must be.

B: *Now consider more generally an annuity that pays* x *every year beginning next year for a period of* n *years when the interest rate is* r. *Denote the value of such an annuity as* $y(x,n,r)$.

(a) *Derive the general formula for* $y(x,n,r)$ *by using the trick described in part A.*

Answer: The value of an infinitely-lasting annuity that pays x per year (starting one year from now) is x/r. The value of such an infinitely-lasting annuity n years from now is

$$\frac{x/r}{(1+r)^n} = \frac{x}{r(1+r)^n}. \tag{3.16}$$

We can then determine the value of the n-year annuity described in the problem as the difference between two infinitely-lasting annuities; i.e.

$$y(x,n,r) = \frac{x}{r} - \frac{x}{r(1+r)^n} = \frac{x\left((1+r)^n - 1\right)}{r(1+r)^n} \tag{3.17}$$

(b) *Apply the formula to the following example: You are about to retire and have $2,500,000 in your retirement fund. You can take it all out as a lump sum, or you can choose to take an annuity that will pay you (and your heirs if you pass away) $\$x$ per year (starting next year) for the next 30 years. What is the least x has to be in order for you to choose the annuity over the lump sum payment assuming an interest rate of 6%.*

Answer: Substituting $n = 30$ and $r = 0.06$ into the formula for the value of the annuity, we get

$$\frac{x\left((1.06)^{30} - 1\right)}{0.06(1.06)^{30}} = 13.76483x. \tag{3.18}$$

The only way you will accept the annuity is if its value is at least $2,500,000; i.e.

$$13.76483x \geq 2,500,000. \tag{3.19}$$

Solving for x, we get $181,622.30. Thus, the annuity has to pay at least this much per year.

(c) *Apply the formula to another example: You can think of banks as accepting annuities when they give you a mortgage. Suppose you determine you would be able to pay at most $10,000 per year in mortgage payments. Assuming an interest rate of 10%, what is the most the bank will lend you on a 30-year mortgage (where the mortgage payments are made annually beginning 1 year from now)?*

Answer: The bank would be willing to lend you

$$\frac{\$10000\left((1+0.10)^{30} - 1\right)}{0.1(1+0.1)^{30}} = \$94,269.14. \tag{3.20}$$

(d) *How does your answer change when the interest rate is 5%?*

Answer: When the interest rate is $r = 0.05$, the formula becomes

$$\frac{\$10000\left((1+0.05)^{30} - 1\right)}{0.05(1+0.05)^{30}} = \$153,724.51. \tag{3.21}$$

(e) *Can this explain how people in the late 1990's and early 2000's were able to finance increased current consumption as interest rates fell?*

Answer: Suppose the bank originally lent you $94,269 at 10% interest with you making $10,000 in mortgage payments per year. Then the interest rate falls to 5% — and you can now re-finance and borrow $153,725 and still only make $10,000 in annual mortgage payments. Thus, you have additional money for consumption by re-financing. This was a major source of increased consumption in the U.S. in the late 1990's and early 2000's as people re-financed their homes given low interest rates — and consumed the additional amount they were able to borrow. (Another big part of the story was that home values were increasing — also allowing consumers to borrow more on their home equity.)

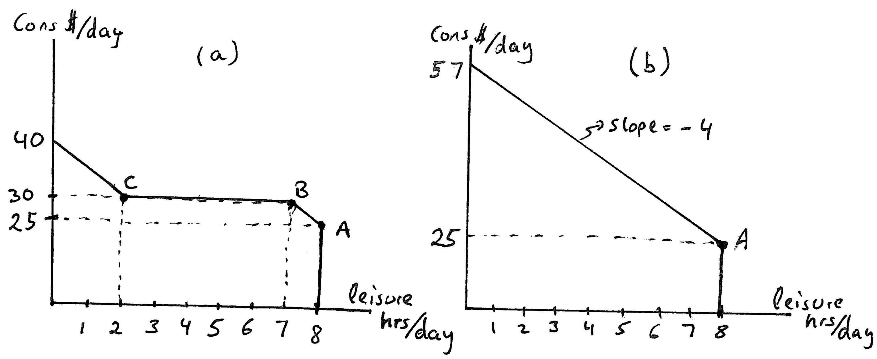
Exercise 3.18

Policy Application: *AFDC versus a Negative Income Tax*: *Until the late 1990's, one of the primary federal welfare programs was AFDC — or Aid to Families with Dependent Children. The program was structured roughly similarly to the following example: Suppose you can work any number of hours you choose at $5* *per hour and you have no income other than that which you earn by working. If you have zero overall income, the government pays you a welfare payment of $25 per day. You can furthermore receive your full welfare benefits so long as you make no more than a total income of $5 per day. For every dollar you earn beyond $5, the government reduces your welfare benefits by exactly a dollar until your welfare benefits go to zero.*

A: *Suppose you have up to 8 hours of leisure per day that you can dedicate to work.*

(a) *Draw your budget constraint between daily leisure and daily consumption (measured in dollars).*

Answer: Panel (a) of Graph 3.10 plots this budget constraint. If begins at bundle *A* with 8 hours of leisure and $25 in welfare benefits. Since welfare benefits do not get reduced for the first $5 earned, one less hour of leisure translates into $30 rather than $25 of consumption — bundle *B* in the graph. But from then on, until all $25 in welfare benefits are exhausted, decreased leisure (i.e. increased work) does not translate into additional consumption. At bundle *C*, all welfare payments are gone and the worker again benefits from reducing his leisure.



Graph 3.10: AFDC versus a Negative Income Tax

(b) *If you define marginal tax rates in this example as the fraction of additional dollars earned in the labor market that a worker does not get to keep, what is the marginal tax rate faced by this worker when he is working 1 hour per day? What if he is working 5 hours per day? What if he is working 6 hours a day?*

Answer: The marginal tax rate faced by a worker who is working 1 hour per day is 1.00, or 100%. This is because welfare benefits are reduced dollar for dollar as additional income is earned in the labor market. The same is true for a worker who works 5 hours per day. Once the worker is working 6 hours per day, the marginal tax rate falls from 100% to zero.

(c) *Without knowing anything about tastes, how many hours are you likely to work under these tradeoffs?*

Answer: It seems likely that a worker will work no more than 1 hour given that he faces a 100% tax rate for the next five hours of work.

(d) *The late Milton Friedman was critical of the incentives in the AFDC program and proposed a different mechanism for supporting the poor. He suggested a program, known as the negative income tax, that works something like this: Everyone is guaranteed $25 per day that he/she receives regardless of how much he/she works. Every dollar from working, starting with the first one earned, is then taxed at* $t = 0.2$. *Illustrate our worker's budget constraint assuming AFDC is replaced with such a negative income tax.*

Answer: Panel (b) of Graph 3.10 graphs the negative income tax budget constraint. It begins at the same point A as the AFDC budget and then increases at a rate of \$4 per hour worked because the 20% tax begins immediately (thus causing the after-tax wage to be \$4 per hour.)

(e) *Which of these systems will almost certainly cost the government more for this worker — the AFDC system or the negative income tax? Which does the worker most likely prefer? Explain.*

Answer: The AFDC system almost certainly costs more. This is because it is likely that the worker will work at most 1 hour under the AFDC system — and will collect the entire \$25 in welfare. Under the negative income tax, on the other hand, it is likely that the worker will work more — and will thus pay some taxes that will reduce the net-payment to him under the negative income tax below \$25. Since almost the entire AFDC budget lies within the negative income tax budget, it is likely that the worker would prefer the negative income tax. (Only a small corner of the AFDC budget around point B would stick out of the negative income tax budget if the two were laid one on top of the other. It seems quite likely that the guaranteed income could be raised sufficiently for the entire AFDC budget to be contained in the negative income tax budget and still the government pays out less under the negative income tax than AFDC.)

(f) *What part of your negative income tax graph would be different for a worker who earns \$10 per hour?*

Answer: The constraint would still begin at A but would go up at a steeper rate — at \$8 per hour worked (given that the before tax wage of \$10 falls to \$8.)

(g) *Do marginal tax rates for an individual differ under the negative income tax depending on how much leisure he/she consumes? Do they differ across individuals?*

Answer: No, the marginal tax rate under the negative income tax is always the same — for any individual regardless of how much she works, and across individuals regardless of how much they make.

B: *Consider a more general version of the negative income tax, one that provides a guaranteed income y and then reduces this by some fraction t for every dollar earned — resulting eventually in individuals with sufficiently high income paying taxes.*

(a) *Derive a general expression for the budget constraint under a negative income tax, a constraint relating daily consumption c (in dollars) to daily leisure hours ℓ assuming that at most 8 hours of leisure are available.*

Answer: $c = y + (1-t)w(8-\ell)$

(b) *Derive an expression for how much the government will spend (or receive) for a given individual depending on how much leisure she takes.*

Answer: The government gives y as guaranteed income but then collects $tw(8-\ell)$ in taxes from this worker. Thus, the overall government payment (or receipt) is $y - tw(8-\ell)$.

(c) *Derive expressions for marginal and average tax rates as a function of daily income I, the guaranteed income level y and the tax rate t.* (Hint*: Average tax rates can be negative.)*

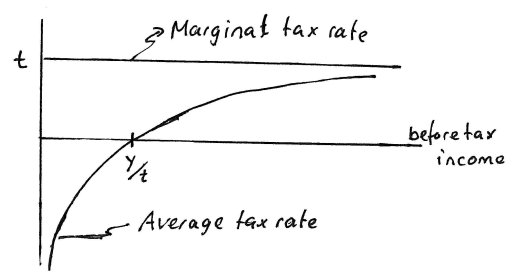
Answer: As we already determined above, the marginal tax rate is constant and equal to t. The average tax rate is the total in net-tax payments divided by income. An individual pays tI in taxes but also collects y — which implies that the total net-payment is $tI - y$. Thus, the average tax rate is $(tI - y)/I$ which is negative for low levels of I.

(d) *On a graph with daily before-tax income on the horizontal axis and tax rates on the vertical, illustrate how marginal and average tax rates change as income rises.*

Answer: Graph 3.11 (on the next page) depicts these, with the average tax rate given by $(tl - y)/I$ as derived above. When $tI = y$, income has risen sufficiently for tax payments to exactly offset the guaranteed income payment. Solved for I, this occurs at $I = y/t$ — which is where the average tax function crosses the intercept because the average tax at that point is zero. The average tax function approaches but never converges to the marginal tax rate t because, no matter how high income gets, everyone receives the guaranteed income y.

(e) *Is the negative income tax progressive?*

Answer: A tax is progressive if the average tax rate increases with income. This is the case here — so yes, the negative income tax is progressive.



Graph 3.11: Average and Marginal Tax Rates under the Negative Income Tax

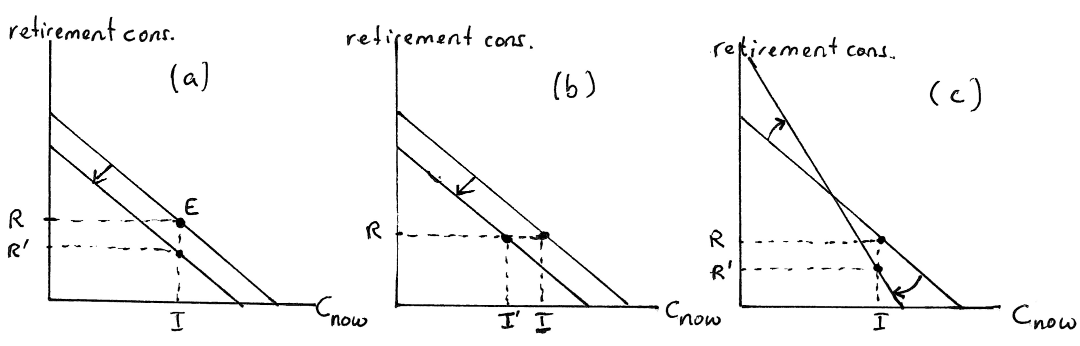
Exercise 3.20

Policy Application: *Three Proposals to Deal with the Social Security Shortfall: It is widely recognized that the social security systems in many western democracies will face substantial shortfalls between anticipated revenues and promised benefits over the coming decades.*

A: *Various ideas have emerged on how we should prepare for this upcoming shortfall.*

(a) *In order to analyze the impact of different proposals, begin with a graph that has "consumption now" on the horizontal and "retirement consumption" on the vertical axes. For simplicity, suppose we can ignore periods between now and retirement. Consider a worker and his choice set over these two "goods". This worker earns some current income I, and he is currently promised a retirement income R from the government. Illustrate how this establishes an "endowment point" in your graph. Then, assuming an interest rate r over the period between now and retirement, draw this worker's choice set.*

Answer: Panel (a) of Graph 3.12 graphs this budget constraint as the one through the endowment point labeled E. The slope is $-(1+r)$.



Graph 3.12: 3 Social Security Reforms

(b) *Some have proposed that we need to cut expected retirement benefits for younger workers — i.e. we need to cut R to $R' < R$. Illustrate the impact this has on our worker's choice set.*

Answer: This change is also graphed in panel (a) of the graph. The policy shifts the expected retirement income but does not alter r and thus does not alter the opportunity cost of consuming now versus consuming later. Thus, the slope stays the same and the movement in the endowment point simply results in a parallel inward shift of the budget constraint.

(c) *Others have argued that we should instead raise social security taxes — i.e. reduce I to $I' < I$ — in order to prepare for the upcoming shortfall. Illustrate how this would impact our worker's budget constraint.*

Answer: This is illustrated in panel (b). This time, the current income I is shifted but the expected social security income R remains unchanged. Again, nothing has changed the interest rate and thus the opportunity cost of consuming now versus later remains unchanged. We again get an inward parallel shift of the budget constraint as the government policy adjusts the endowment point.

(d) *Assuming that r is not impacted differently by these two policies, could you argue that they are essentially the same policy?*

Answer: Both result in a parallel inward shift of budgets. So, as long as the size of the reduction in R on the one hand and of I on the other is comparable, the two policies are identical in their impact on choice sets.

(e) *Yet others have argued that we should lower future retirement benefits R but at the same time subsidize private savings — i.e. increase r — through policies like expanding tax deferred savings accounts. Illustrate the impact of lowering R and raising r.*

Answer: This is illustrated in panel (c) of Graph 3.12 where the increase in r causes the slope of the budget to become steeper while the decrease in R causes the endowment point to shift down.

(f) *Which of these policies is the only one that has a chance (though by no means a guarantee) of making some individuals better off?*

Answer: Only the last one might make some individuals better off — because only the last policy results in a new budget constraint that might contain some points that were not contained in the original budget constraint.

B: *Define I, R and r as above.*

(a) *Write down the mathematical description of the current intertemporal budget for our worker — in terms of I, R and r. Let c_1 denote current consumption and let c_2 denote retirement consumption.*

Answer: Retirement consumption will depend on how much we save $(I - c_1)$ plus the level of social security benefits R that we receive. The budget constraint is therefore

$$c_2 = (1+r)(I - c_1) + R = (1+r)I + R - (1+r)c_1. \tag{3.22}$$

(b) *In your equation, show which parts correspond to the vertical intercept and slope in your graphs from part A.*

Answer: The intercept term is $[(1+r)I + R]$ and the slope term is $-(1+r)$.

(c) *Relate your equation to the changes that you identified in the graph from each of the policies.*

Answer: When only R is changed (as in panel (a) of Graph 3.12), only the intercept term is affected since R appears there but not in the slope term. The same is true when I is changed (as in panel (b) of Graph 3.12). Finally, when both R and r are changed, both the intercept and slope terms are affected. Whether the new intercept term is higher than the old one (as in the graph) or lower depends on the relative magnitude of the change in R and r.

Conclusion: Potentially Helpful Reminders

1. When illustrating either worker or intertemporal budget constraints, always be sure you know where the endowment bundle lies — because the constraint will always rotate through that bundle as wages or interest rates change.

2. If you are unsure where the endowment bundle lies, just ask yourself: Which bundle can the worker (or saver or borrower) consumer *regardless* of what wages or interest rates are?

3. When leisure is modeled on the horizontal axis, then the slope of the worker's budget is $-w$ — or minus the wage. When current consumption is modeled on the horizontal axis of an intertemporal budget constraint, the slope of the budget line is $-(1+r)$

4. If you are interested in finance applications, check out in particular the end-of-chapter exercises 3.9 through 3.14.

5. End-of-chapter exercises 3.15 takes you through illustrating tax revenue in the worker budget graph — a skill that will show up repeatedly throughout the text, beginning in Chapter 8.